EPFL

Chapter 3: One-parameter groups (Lie groups)

Similarity and Transport Phenomena in Fluid Dynamics

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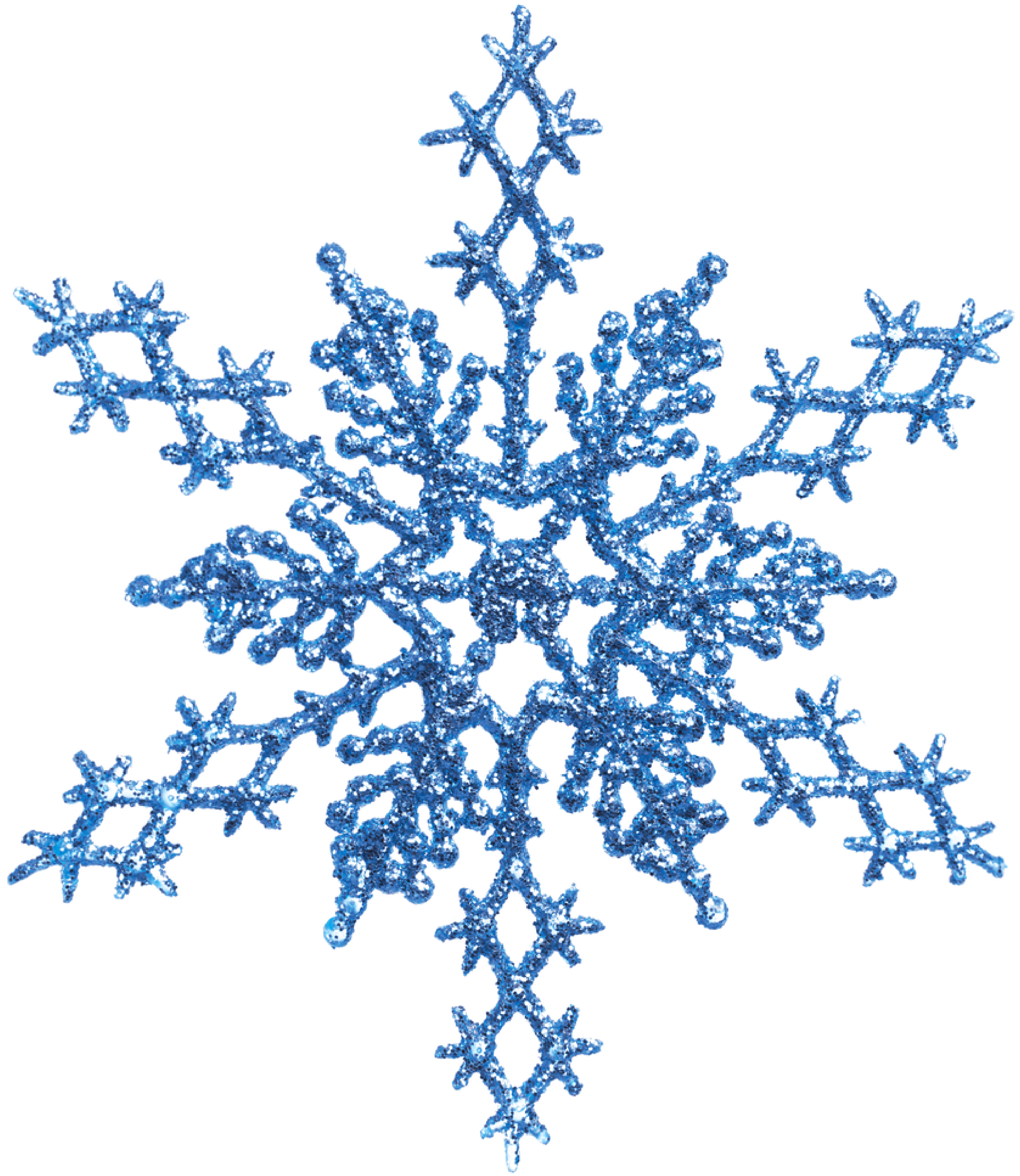
Chapter 3: One-parameter groups



- Groups of transformation
- Infinitesimal transformation
- Group invariants
- Invariant curves
- Transformation of derivative

Groups of transformation

Consider the transformation

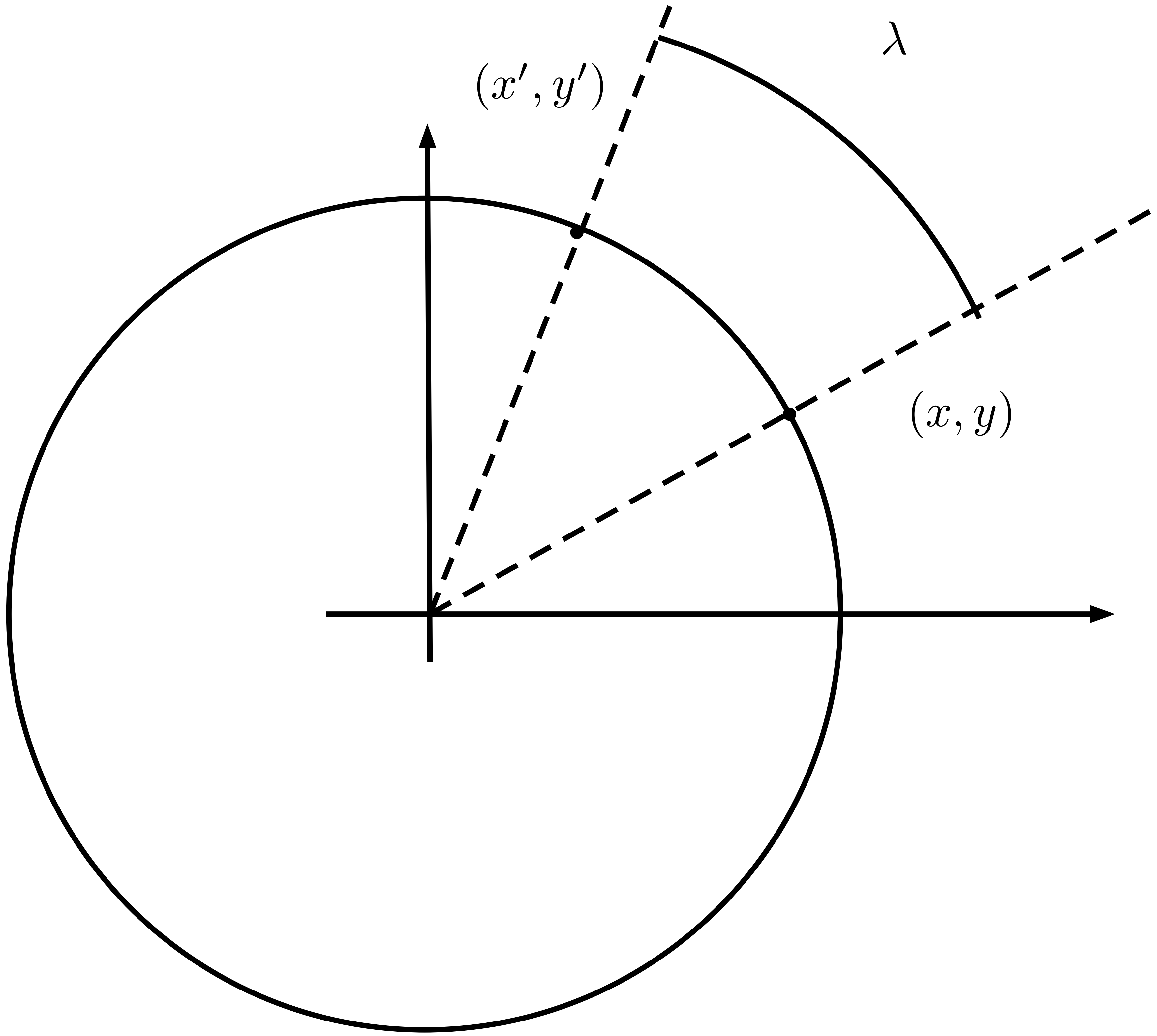
$$x' = X(x, y; \lambda),$$
$$y' = Y(x, y; \lambda),$$

that depends on the continuous parameter λ. For instance, the rotation around the origin

$$x' = x \cos \lambda - y \sin \lambda,$$
$$y' = x \sin \lambda + y \cos \lambda,$$

where λ is the angle of rotation.

Groups of transformation



Properties of rotations: group properties

- two rotations λ and λ' carried out in succession are equivalent to the rotation $\lambda'' = \lambda + \lambda'$
- If $\lambda' = -\lambda$, then the second image coincides with the source point (*identity transformation*)
- each rotation has an inverse

Infinitesimal transformation

λ_0 the value of λ for which the transformation is the identity transformation. First order Taylor's series around λ_0

$$x' = X(x, y; \lambda) = x + \left.\frac{\partial X}{\partial \lambda}\right|_{\lambda_0} (\lambda - \lambda_0) + \cdots \ ,$$

$$y' = Y(x, y; \lambda) = y + \left.\frac{\partial Y}{\partial \lambda}\right|_{\lambda_0} (\lambda - \lambda_0) + \cdots \ ,$$

We introduce the coefficients of the infinitesimal transformation:

$$\xi = \left.\frac{\partial X}{\partial \lambda}\right|_{\lambda_0} \quad \text{and} \quad \eta = \left.\frac{\partial Y}{\partial \lambda}\right|_{\lambda_0} \ .$$

Working with the group is equivalent to working with its infinitesimal representation (Lie theorem).

Infinitesimal transformation

First-order expansion $=$ infinitesimal transformation

$$x' = x + \xi(x, y)(\lambda - \lambda_0),$$
$$y' = y + \eta(x, y)(\lambda - \lambda_0)$$

This is the Euler approximation of the coupled differential equations

$$\frac{\mathrm{d}x}{\xi(x, y)} = \frac{\mathrm{d}y}{\eta(x, y)} = \mathrm{d}\lambda.$$

The path originating from the source point (x, y) is the locus of all the images of the source point. It is called the *orbit* of the group. The differential operator

$$\Gamma = \xi\partial_x + \eta\partial_y$$

is called the *infinitesimal generator* or the *group operator*.

Exercises 1 and 2

1. Determine the orbit of the rotation group.
2. Does the translation $x' = x + s$ form a group? If so, determine the infinitesimal operator.

Group invariant

A *group invariant* is a function $u(x, y)$ whose value at an image point is the same as its value at the source point

$$u(x', y') = u(x, y) \text{ or } u(X(x, y; \lambda), Y(x, y; \lambda)) = u(x, y).$$

Thus u is constant along an orbit. Differentiating with respect to λ, then setting $\lambda = \lambda_0$, we find

$$\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} = 0 \Leftrightarrow \frac{\mathrm{d}x}{\xi(x, y)} = \frac{\mathrm{d}y}{\eta(x, y)} \Leftrightarrow \Gamma u = 0.$$

Any function of the integral of this characteristic equation is an invariant.

Invariant curves

An *invariant curve* is one whose points map into other points for all transformations of the group. The curve C is either an orbit of the group or a locus on which the infinitesimal coefficients ξ and η vanish simultaneously. An implicit representation of curves is

$$\phi(x, y) = c,$$

where ϕ is a one-parameter family of curves and c is a parameter. The family is invariant if the image of each curve is another curve of the family, namely

$$\phi(x', y') = c'$$

Invariant curves

Let us differentiate $\phi(x', y') = c'$ with respect to λ and set $\lambda = \lambda_0$

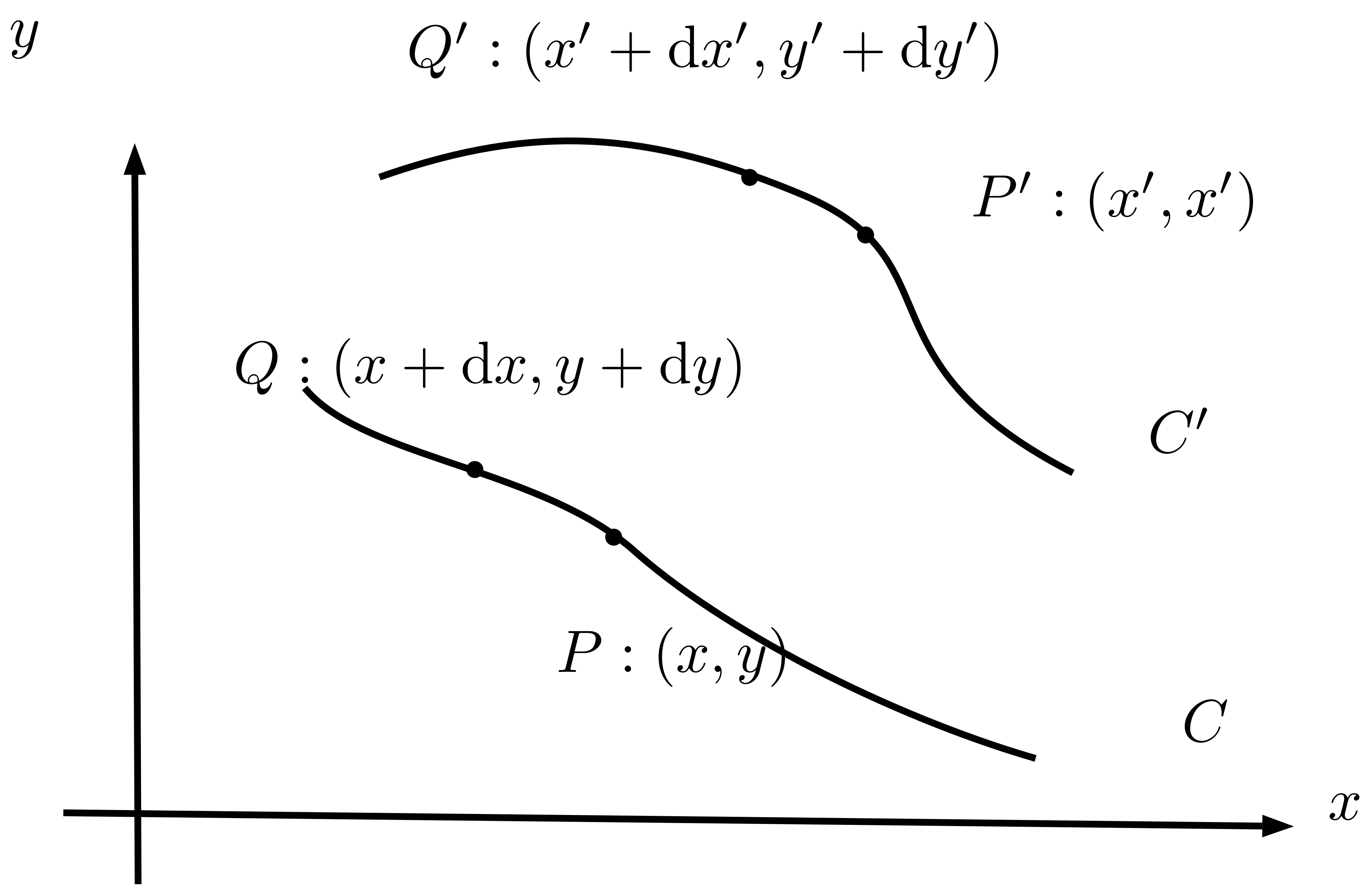
$$\xi \frac{\partial \phi}{\partial x} + \eta \frac{\partial \phi}{\partial y} = \left. \frac{\partial c'}{\partial \lambda} \right|_{\lambda = \lambda_0}.$$

The right-hand side is a function of c alone (let us call it F). Its choice is up to us (we then obtain different families). For instance, we can take $F = 1$.

Summary:

- $\Gamma u = 0$ is the equation for invariant curves.
- $\Gamma u = 1$ is the equation of invariant families.

Extended group



Question: $C \to C'$, how to calculate $\dot{y}' = \mathrm{d}y'/\mathrm{d}x'$ from $\dot{y} = \mathrm{d}y/\mathrm{d}x$?

Coordinates of P'

$$x' = X(x, y; \lambda)$$

$$y' = Y(x, y; \lambda)$$

and those of Q'

$$x' + \mathrm{d}x' = X(x + \mathrm{d}x, y + \mathrm{d}y; \lambda)$$

$$y' + \mathrm{d}y' = Y(x + \mathrm{d}x, y + \mathrm{d}y; \lambda)$$

Extended group

To leading order, we get

$$\mathrm{d}x' = X_x \mathrm{d}x + X_y \mathrm{d}y$$
$$\mathrm{d}y' = Y_x \mathrm{d}x + Y_y \mathrm{d}y$$

whose ratio is

$$\dot{y}' = \frac{\mathrm{d}y'}{\mathrm{d}x'} = \frac{Y_x \mathrm{d}x + Y_y \mathrm{d}y}{X_x \mathrm{d}x + X_y \mathrm{d}y}.$$

The transformation $(x, y, \dot{y}) \to (x', y', \dot{y}')$ forms a group called the *once-extended group*. What is the coefficient of the infinitesimal transformation?

Extended group

Using the infinitesimal form, we have

$$\mathrm{d}x' = \mathrm{d}x + \mathrm{d}\xi(\lambda - \lambda_0)$$
$$\mathrm{d}y' = \mathrm{d}y + \mathrm{d}\eta(\lambda - \lambda_0)$$

We then find that the slope of the image curve is

$$\dot{y}' = \frac{\mathrm{d}y'}{\mathrm{d}x'} = \frac{\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{\mathrm{d}\eta}{\mathrm{d}x}(\lambda - \lambda_0)}{1 + \frac{\mathrm{d}\xi}{\mathrm{d}x}(\lambda - \lambda_0)}.$$

To leading order, we find

$$\dot{y}' = \frac{\mathrm{d}y'}{\mathrm{d}x'} = \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}\eta}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)(\lambda - \lambda_0).$$

Extended group

The coefficient is then

$$\boxed{\eta_1 = \frac{\mathrm{d}\eta}{\mathrm{d}x} - \dot{y}\frac{\mathrm{d}\xi}{\mathrm{d}x}.}$$

with the total derivatives (directional derivatives in the direction $\dot{y}$)

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \eta_x + \eta_y \dot{y}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = \xi_x + \xi_y \dot{y}$$

Higher-order derivatives are obtained by iteration ($k \geq 1$)

$$\boxed{\eta_{k+1} = \frac{\mathrm{d}\eta_k}{\mathrm{d}x} - y^{(k+1)}\frac{\mathrm{d}\xi}{\mathrm{d}x}}$$

Invariant differential equations of the first order

An invariant of the once-extended group is a function $u(x, y, \dot{y})$ that satisfies

$$u(x, y, \dot{y}) = u(x', y', \dot{y}')$$

Upon differentiation, we show that u satisfies

$$\xi u_x + \eta u_y + \eta_1 u_{\dot{y}} = 0,$$

or

$$\frac{\mathrm{d}x}{\xi(x,y)} = \frac{\mathrm{d}y}{\eta(x,y)} = \frac{\mathrm{d}\dot{y}}{\eta_1(x,y,\dot{y})}.$$

This extension has interesting applications that we will see in the next chapters.

Exercise 3

3. Consider the stretching group

$$x' = \lambda x,$$
$$y' = \lambda^{\beta} y$$

Determine the coefficients of the infinitesimal transformation and the orbit.