Chapter 6: Similarity solutions of partial differential equations

Similarity and Transport Phenomena in Fluid Dynamics

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Chapter 6: Similarity solutions of partial differential equations

- Similarity solutions
- Associated stretching groups
- Asymptotic behaviour
- Case studies

Stretching groups

Let us consider a partial differential equation in the form

$$f\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial x}, \dots\right) = 0.$$

We study the existence and properties of similarity solutions. Not all solutions to PDEs are similarity solutions, PDEs do not always have similar solutions, but when they exist, they shed light on the behaviour of more general solutions. We consider the one-parameter stretching group

$$x' = \lambda x, t' = \lambda^{\beta} t, \text{ and } c' = \lambda^{\alpha} c$$

with λ the group parameters, and α and β the family parameters of the group that label different groups of the family. The values of α and β are usually fixed by the initial and boundary conditions.

Stretching groups: similarity solutions

The solution to the PDE is a surface in the (x, t, c) space. Its general equation is

$$F(x, t, c) = 0$$

The equation is invariant to the stretching group

$$F(\lambda x, \lambda^{\beta} t, \lambda^{\alpha} c) = 0$$

Differentiating it with respect to λ, then setting $\lambda = 1$ leads to

$$xF_x + \beta t F_t + \alpha c F_c = 0$$

whose characteristic form is

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{\beta t} = \frac{\mathrm{d}c}{\alpha c} = \frac{\mathrm{d}F}{0}$$

Stretching groups: similarity solutions

Three integrals of this equation are

$$\zeta = \frac{x}{t^{1/\beta}}, f = \frac{c}{t^{\alpha/\beta}} \text{ and } F$$

The most general solution is $F = \mathcal{F}(\zeta, f)$ where $\mathcal{F}$ is an arbitrary function. Since $F = 0$, we obtain an explicit solution in the form

$$\boxed{c = t^{\alpha/\beta} f(\zeta) \text{ with } \zeta = \frac{x}{t^{1/\beta}}}$$

When this form is substituted into the PDE, an ODE results. This ODE (called the *principal equation*) is usually simpler to solve than the original PDE.

Asymptotic behaviour of similarity solutions

If the initial and boundary conditions impose that $M\alpha + N\beta = L$ (with M, N, and L constants), then the ODE is invariant to the *associated stretching group*

$$\zeta' = \lambda\zeta \text{ and } f' = \lambda^{L/M} f$$

If $L/M < 0$, then the principal equation admits an asymptotic solution $f = A\zeta^{L/M}$. The value of A is found by substituting this form into the PDE.

(Note: usually in fluid mechanics, but not always, we are looking for solutions that represent spreading of a material/quantity, this is why we are interested in $L/M < 0$.)

Exercises 1 and 2

1. Consider the linear diffusion equation:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

with D the diffusivity. Mass conservation implies that $\int_{\mathbb{R}} c(x,t)\mathrm{d}x = c_0$. Show that this PDE is invariant to a stretching group. Solve the problem.

2. Consider the same linear diffusion equation subject to the boundary conditions $c(0,t) = \sin(\omega t)$. Does the problem admit a similarity solution?

Exercise 3: the dam-break problem

3. Consider the shallow water equations:

$$\partial_t h + \partial_x (uh) = 0,$$

$$\partial_t u + u\partial_x u + g\partial_x h = 0,$$

subject to

for u,	$-\infty < x < \infty$	$u(x,0) = 0$
for h,	$x > 0$	$h(x,0) = h_0$
	$x < 0$	$h(x,0) = 0$

Show that these PDEs admit a similarity solution and determine this solution.

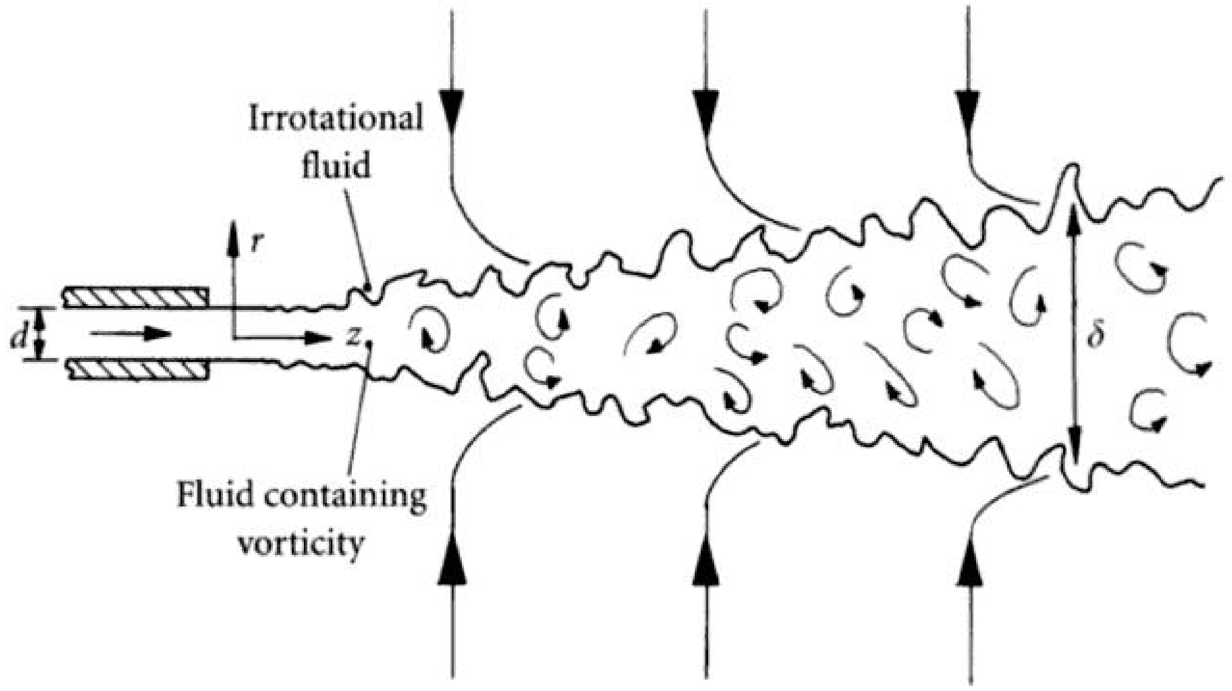
Ritter, A., Die Fortpflanzung der Wasserwellen, *Zeitschrift des Vereines Deutscher Ingenieure*, 36 (33), 947-954, 1892.

Exercise 4: unsteady impulsive flows

4. We address the *Stokes' first problem* (also called *Rayleigh problem*). Consider an incompressible Newtonian fluid bounded by an upper solid boundary. At $t = 0$, the plate is suddenly and instantaneously set into motion. Its velocity is $U(t) = At^{\alpha}$ with A a constant and $\alpha > 0$. The fluid velocity adjusts to this change in the boundary condition. Write the Navier-Stokes equations and the boundary conditions. Reduce the equations and determine their type (hyperbolic, parabolic, elliptic). Seek similarity solutions.

Drazin, P.G., and N. Riley, *The Navier-Stokes Equations: A classification of Flows and Exact Solutions*, Cambridge University Press, Cambridge, 2006.

Exercise 5: round jet



5. We consider a steady turbulent round jet. There is a dominant flow direction z, and the flow spreads gradually in the r direction. The axial gradients are assumed small compared with lateral gradients. These features allow boundary-layer equations to be used in place of the Reynolds equation (even in the absence of a boundary).

Pope, S.B., *Turbulent Flows*, 771 pp., Cambridge University Press, Cambridge, 2000.

Exercise 5: round jet

We use a simple closure equation for the velocity cross-correlation

$$\langle u'v' \rangle = -\nu_t \frac{\partial \langle u \rangle}{\partial z}$$

with $\nu_t \sim 0.028$. The momentum balance equation reduces to

$$\langle u \rangle \frac{\partial \langle u \rangle}{\partial z} + \langle v \rangle \frac{\partial \langle u \rangle}{\partial r} = \frac{\nu_t}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \langle u \rangle}{\partial r} \right)$$

subject to the boundary conditions $u = v = 0$ along the z-axis and $u = v = 0$ in the limit $r \to \infty$. Introduce the stream function ψ ($\langle u \rangle = r^{-1} \psi_r$ and $\langle v \rangle = -r^{-1} \psi_z$). Find a similarity solution to the momentum balance equation.

Exercise 6: front propagation in viscoplastic fluids

6. Let us consider a viscoplastic material. In time-dependent simple-shear flow, the momentum balance equation reduces to

$$\varrho \frac{\partial u}{\partial t} = \frac{\partial}{\partial z}(\mu(\dot{\gamma})\dot{\gamma}) \text{ with } \dot{\gamma} = \frac{\partial u}{\partial z}$$

where μ is the bulk viscosity given by Carreau's model

$$\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1 + (\gamma/\gamma_c)^2)^{(1-n)/2}}$$

with n the shear-thinning/thickening index, μ_∞ viscosity in the high shear-rate limit, μ_0 viscosity in low high shear-rate limit, and $\dot{\gamma}_c$ a constant.

Duffy, B.R., D. Pritchard, and S.K. Wilson, The shear-driven Rayleigh problem for generalised Newtonian fluids, *J. Non-Newt. Fluid Mech.*, 206, 11-17, 2014.

Exercise 6: front propagation in viscoplastic fluids

The nonlinear diffusion equation is subject to the boundary condition

$$\tau = \mu\dot{\gamma} = \tau_0 \text{ at } z = 0 \text{ and } u \to 0 \text{ when } z \to -\infty$$

Show that the equation is invariant to a similarity group. Under which conditions, is there a front propagating downward? Hint: Show that $u = t^a f(\xi)$ with ξ the similarity variable to be determined. Then consider two possible behaviours for f: (i) a boundary-layer approximation $f \sim A(\xi - \xi_f)^k$ when $\xi \to \xi_f^+$ (ξ_f front position), (ii) algebraic decay in the far field $f \sim B(-\xi)^p$ when $\xi \to -\infty$.

Exercise 7: propagation of a gravity current

7. Consider the shallow equations for a gravity current propagating over a horizontal boundary

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g'\frac{\partial h}{\partial x} = 0.$$

with $g' = g(\varrho - \varrho_a)/\varrho$ the reduced gravity constant, ϱ_a the ambient fluid's density.

Rottman, J.W., and J.E. Simpson, Gravity currents produced by instantaneous releases of a heavy fluid in a rectangular channel, *J. Fluid Mech.*, 135, 95-110, 1

Gratton, J., and C. Vigo, Self-similar gravity currents with variable inflow revisited: plane currents, *J. Fluid Mech.*, 258, 77-104, 1994.

Exercise 7: propagation of a gravity current

The gravity is supplied from a source located at $x = 0$, with an inflow rate

$$Q(t) = uh|_{x=0} = \alpha q t^{\alpha-1},$$

with $\alpha \geq 0$. The Froude number at the source is fixed

$$\mathrm{Fr}|_{x=0} = \frac{u_0}{\sqrt{g' h_0}} F_0,$$

with F_0 a constant. The front takes the form of a blunt nose (shock), with a Benjamin-like condition that related the flow depth to the front velocity

$$\dot{x}_f^2 = \beta^2 g h_f,$$

Show that the governing equations are invariant to a stretching group.

How to study the phase portrait?

Exercise 8: nonlinear diffusion equation

8. Consider the nonlinear diffusion equation

$$\frac{\partial h(x,\ t)}{\partial t} = \frac{\partial}{\partial x}\left(h^n \frac{\partial h(x,\ t)}{\partial x}\right)$$

with the auxiliary equation (mass conservation)

$$M = \int_0^{x_f(t)} h(x,\ t)\mathrm{d}x = A,$$

and

$$h(x_f, t) = 0$$

at the front x_f. Study the phase portrait and deduce the late-time behaviour.

Grundy, R.E., Similarity solutions of the nonlinear diffusion equation, *Quarterly of Applied Mathematics*, 79, 259-280, 1979.

Exercise 9: heat diffusion in a cylindrical geometry

9. In industry, coaxial cylinder viscometers are used to determine the rheological properties of fluid samples. Let us imagine that you need to study the thermal coupling by applying a constant heat flux Q at the inner cylinder $r = R$. What is the resulting temperature at this surface? To simplify the problem we assume that the gap is very large (the outer cylinder is placed at infinity). The governing equation is the heat equation

$$\frac{\partial T}{\partial t} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

with D the diffusivity.

Exercise 9: heat diffusion in a cylindrical geometry

The boundary conditions are

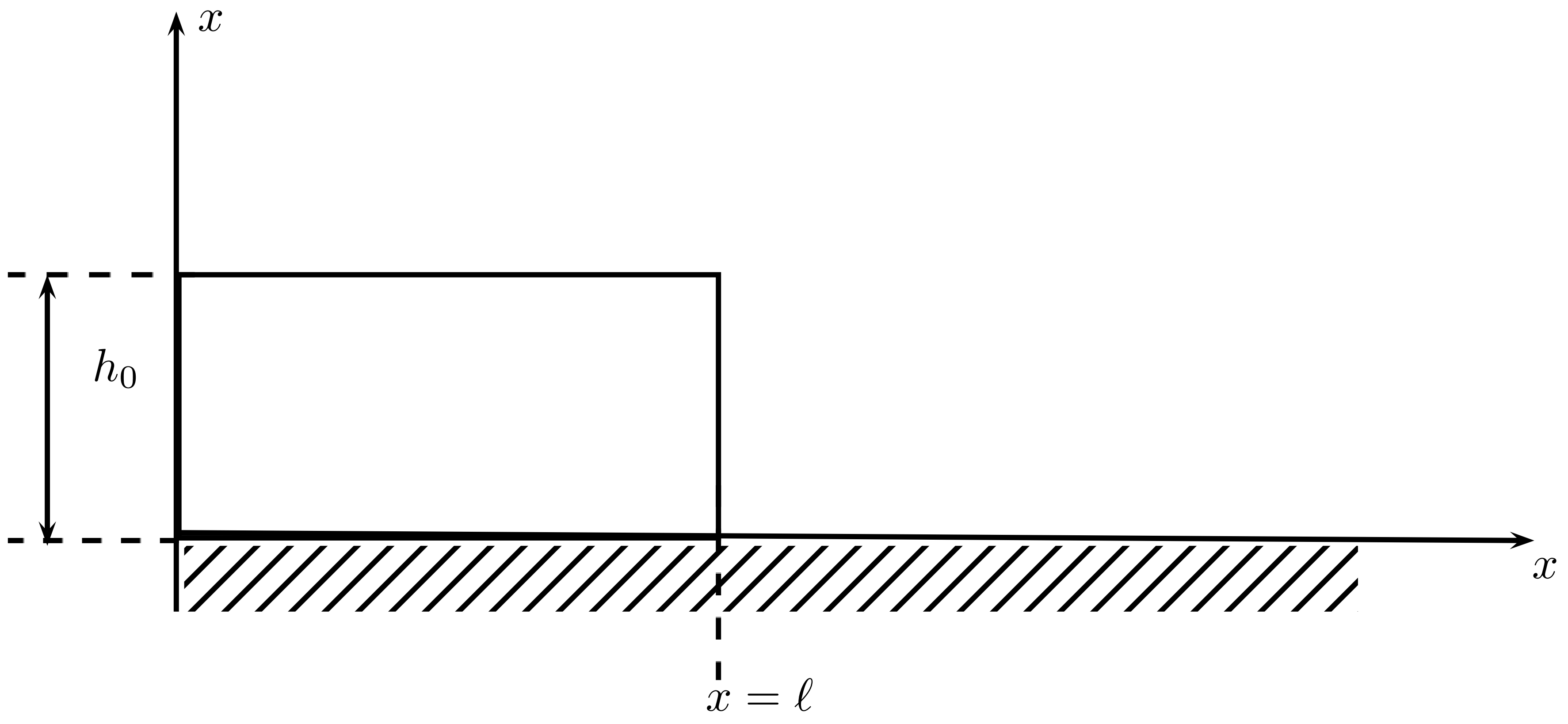
$$\frac{\partial T}{\partial r}(R,t) = -Q \text{ and } T(\infty, r) = 0$$

and the initial condition is

$$T(r,0) = 0$$

- Show that the PDE is invariant to a stretching group.
- What is the problem with the boundary conditions?
- Seek an approximate solution at short times by considering that the heat has not diffused very far.
- Seek an approximate solution at long times by assuming that the heat has spread very far from $r = R$ and so we can take $R = 0$.

Homework: problem statement



Initially, the fluid is contained in a reservoir.

Huppert, H.E., The propagation

two-dimensional and axisymmetric viscous gravity currents

Journal of Fluid Mechanics, 121, 43-58, 1982.

Solve Huppert's equation, which describes fluid motion over a horizontal plane in the low Reynolds-number limit:

$$\frac{\partial h}{\partial t} - \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = 0.$$

The solution must also satisfy the mass conservation equation

$$\int h(x,t)\mathrm{d}x = V_0$$

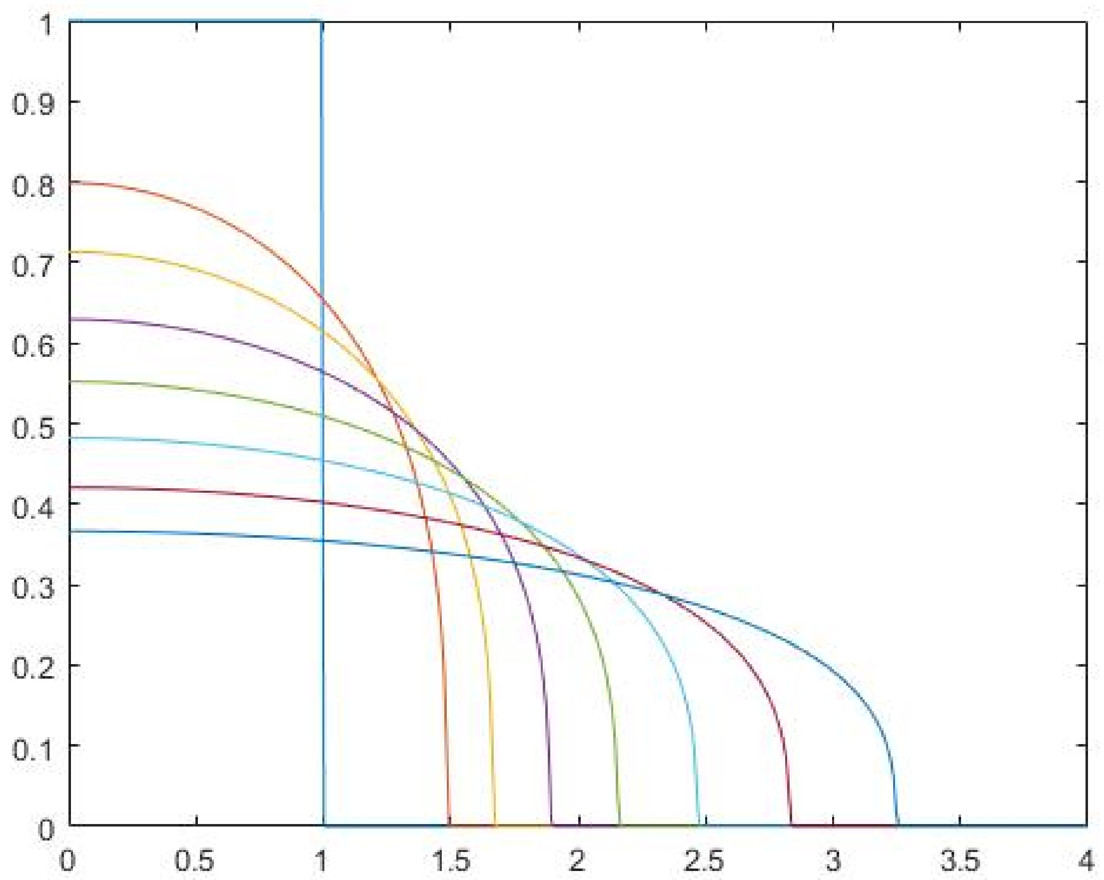
where V_0 is the initial volume $V_0 = \ell h_0$. Calculate the front position with time and show that it varies as $t^{1/5}$. Solve the equation numerically and compare with the analytical solution.

Homework: using pdepe in Matlab

Hint: Show that the PDE is invariant to a stretching group. Solve the principal equation. Use the Matlab built-in function *pdepe* to solve the equation numerically. You can also use an implicit scheme, e.g. Crank-Nicolson. (Scripts on the website).

```
volume = 0.1;
temps = [0:0.1:1 2:0.5:20 21:5:40 40:10:100 200:100:1000];
[x,t,h] = newton(0.1,10,2500,temps);
figure
for i = 1:length(t)
hold on
plot (x,h(i,:))
end
hold off
```

Homework: using pdepe in Matlab




```
function [x,t,h] = newton(hg,xmax,nPoints,t)
m = 0; xmin=0;
x = linspace(xmin,xmax,nPoints);
options = odeset('InitialStep',1e-12);
sol =
 pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t,options,hg);
h = sol(:,:,1);
function [c,f,s] = pdex1pde(x,t,h,DhDx,hg)
c = 1; f = h^ 3 DhDx ; s = 0;
function h0 = pdex1ic(x,hg)
d=0.05;
for i = 1:length(x)
y = x;
if y(i) <= -d/2
h0(i) = 1;
elseif y(i) > -d/2 & y(i) < d/2
h0(i) = ((cos((y(i)-(-d/2))*2*pi/(2*d)))/2+0.5);
else
h0(i) = 0;
end
end
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t,hg)
pl = ul-1; ql = 0;
pr = ur; qr = 0;
```

Homework: using a Crank-Nicolson scheme

In implicit schemes, the spatial derivative is discretized at time $t = k\delta$ and $t = (k+1)\delta$

$$\frac{\partial^2 h}{\partial x^2} = \frac{r}{\delta x^2}(h_{i+1}^{k+1} + h_{i-1}^{k+1} - 2h_i^{k+1}) + \frac{1-r}{\delta x^2}(h_{i+1}^{k} + h_{i-1}^{k} - 2h_i^{k}) + o(\delta x^2)$$

Taking $r = 0.5$ gives the Crank-Nicolson (or Adams-Moulton) scheme.

The scheme is unconditionally stable. For nonlinear diffusion terms, we write

$$\frac{\partial}{\partial x}\left(f(h)\frac{\partial h}{\partial x}\right) = \frac{r}{\delta x^2}\left(f_{i+1/2}^{k+1}(h_{i+1}^{k+1} - h_i^{k+1}) - f_{i-1/2}^{k+1}(h_i^{k+1} - h_{i-1}^{k+1})\right) + \frac{1-r}{\delta x^2}\left(f_{i+1/2}^{k}(h_{i+1}^{k} - h_i^{k}) - f_{i-1/2}^{k}(h_i^{k} - h_{i-1}^{k})\right)$$

with $f_{i+1/2}^{k+1} = (f_{i+1}^{k+1} + f_i^{k+1})/2$