Chapter 7: Travelling wave solutions

Similarity and Transport Phenomena in Fluid Dynamics

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Chapter 7: Travelling wave solutions

- Translation groups
- Example: diffusion with source
- Propagation velocity
- Approach to travelling waves

Translation groups

Translation groups take the form

$$t' = t + \lambda, x' = x + \alpha\lambda \text{ and } c' = c$$

with λ the group parameter and α a family parameter. Solutions that are invariant to a translation group are called *travelling-wave solutions* as physically they can be interpreted as propagating waves.

The focus of this chapter is on the one-dimensional diffusion equation with a source term

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + Q(c)$$

with D the diffusivity and Q the source term.

Translation groups: general solution

A solution that is invariant to a translation group satisfies

$$c(x + \alpha\lambda, t + \lambda) = c(x, t)$$

If we differentiate this equation with respect to λ and set $\lambda = 0$, we obtain

$$\frac{\partial c}{\partial t} + \alpha \frac{\partial c}{\partial x} = 0$$

whose characteristic equations are

$$\frac{\mathrm{d}x}{\alpha} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}c}{0}$$

The two independent integrals are $\zeta = x - \alpha t$ and c. The most general solution is thus $c = C(\zeta)$.

Example: linear diffusion with source

Let us consider the linear diffusion equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + Q(c)$$

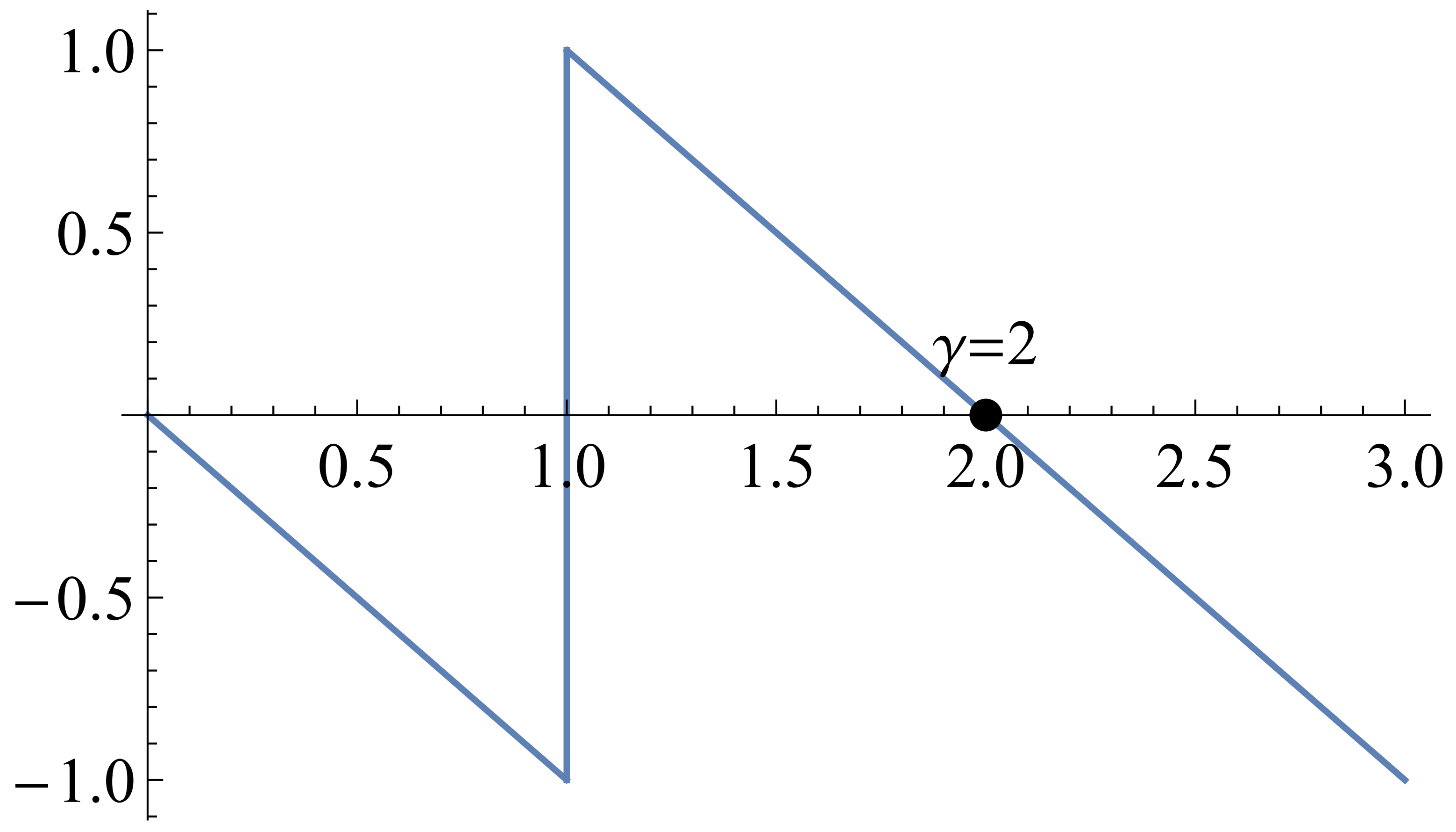
It is invariant to translation groups. Making the change of variable $c = C(\zeta)$ with $\zeta = x - \alpha t$, we obtain the *principal differential equation*

$$D\ddot{C} + \alpha\dot{C} + Q(C) = 0$$

Note that this equation is also invariant to the (associated) translation group $(\zeta, C) \to (\zeta', C') = (\zeta + \mu, C)$. So $u = \dot{C}$ is a first differential invariant. We can transform the principal differential equation into a first-order ODE

$$u = \dot{C} \text{ and } \dot{u} = \ddot{C} = -\frac{\alpha}{D}\dot{C} - \frac{Q(C)}{D} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}C} = -\frac{\alpha}{D} - \frac{Q(C)}{Du}$$

Example: determination of the propagation velocity



Function Q for $\gamma = 2$ and $D = 1$: Note that there are 3 roots: $c = 0$, $c = 1$, and $c = \gamma$. We will mainly work in the fourth quadrant (the solution is expected to satisfy $u \leq 0$, $u(0) = u(\gamma) = 0$)

Solutions in the form $c = C(x - \alpha t)$ represent wave propagation, with α the velocity at which the travelling wave propagates in the x direction. Let us now consider a particular case in which the source terms takes the following form (heat diffusion in superconductors cooled with liquid helium)

$$Q(c) = -c \text{ for } 0 \leq c < 1$$

$$Q(c) = \gamma - c \text{ for } 1 \leq c$$

with $\gamma > 1$ a constant.

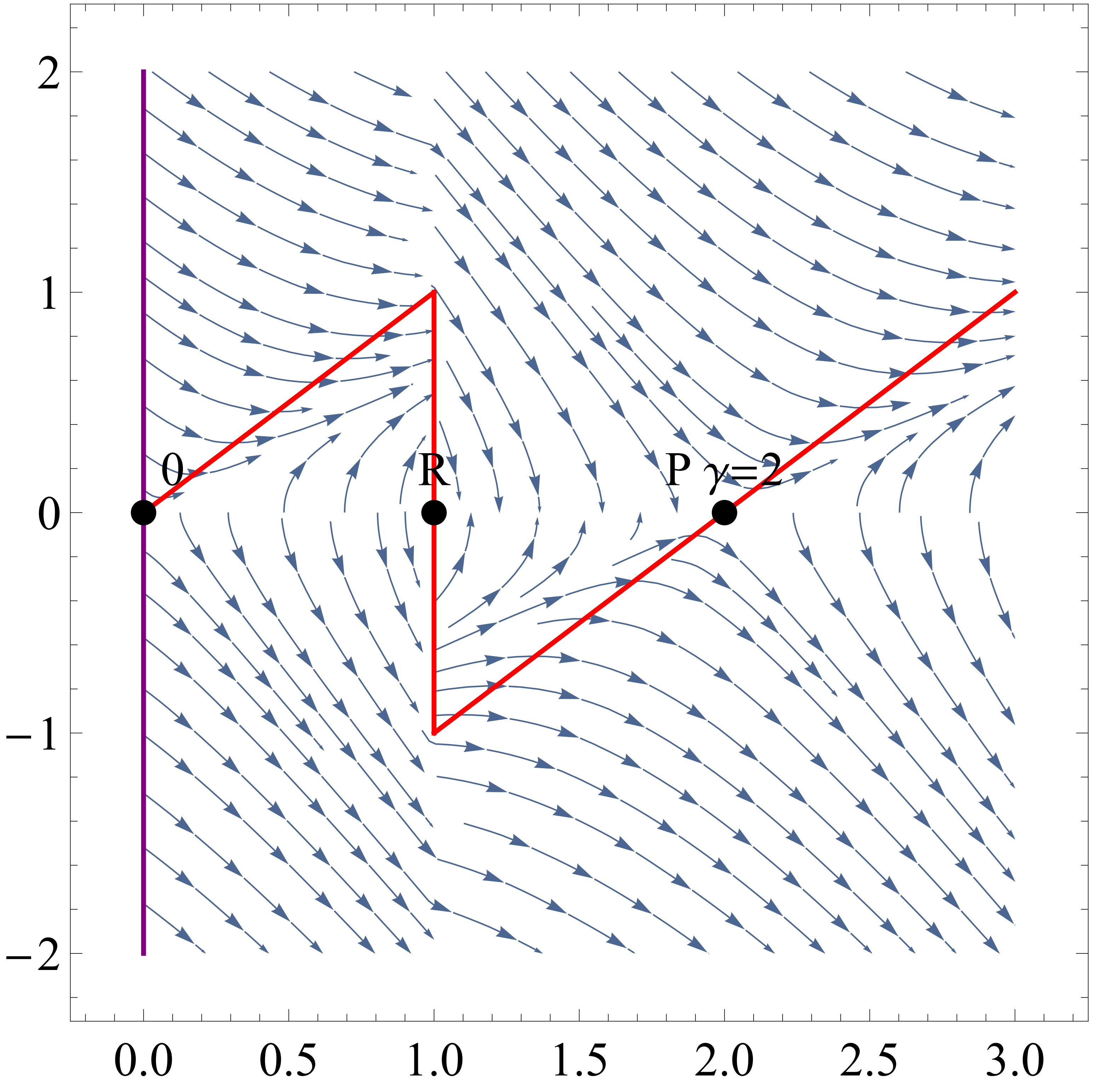
Exercises 1 to 3

Exercise 1. Show that the first and third roots are stable against small perturbations whereas the second is unstable.

Exercise 2. Plot the phase portrait. Determine the qualitative behaviour of the solutions close to the critical points.

Exercise 3. What are the steady states of the system?

Example: determination of the propagation velocity



Phase portrait for $a = 1$, $D = 1$ and $\gamma = 2$

We now calculate the separatrix connecting O to P.

Let us consider the case $C < 1$, then

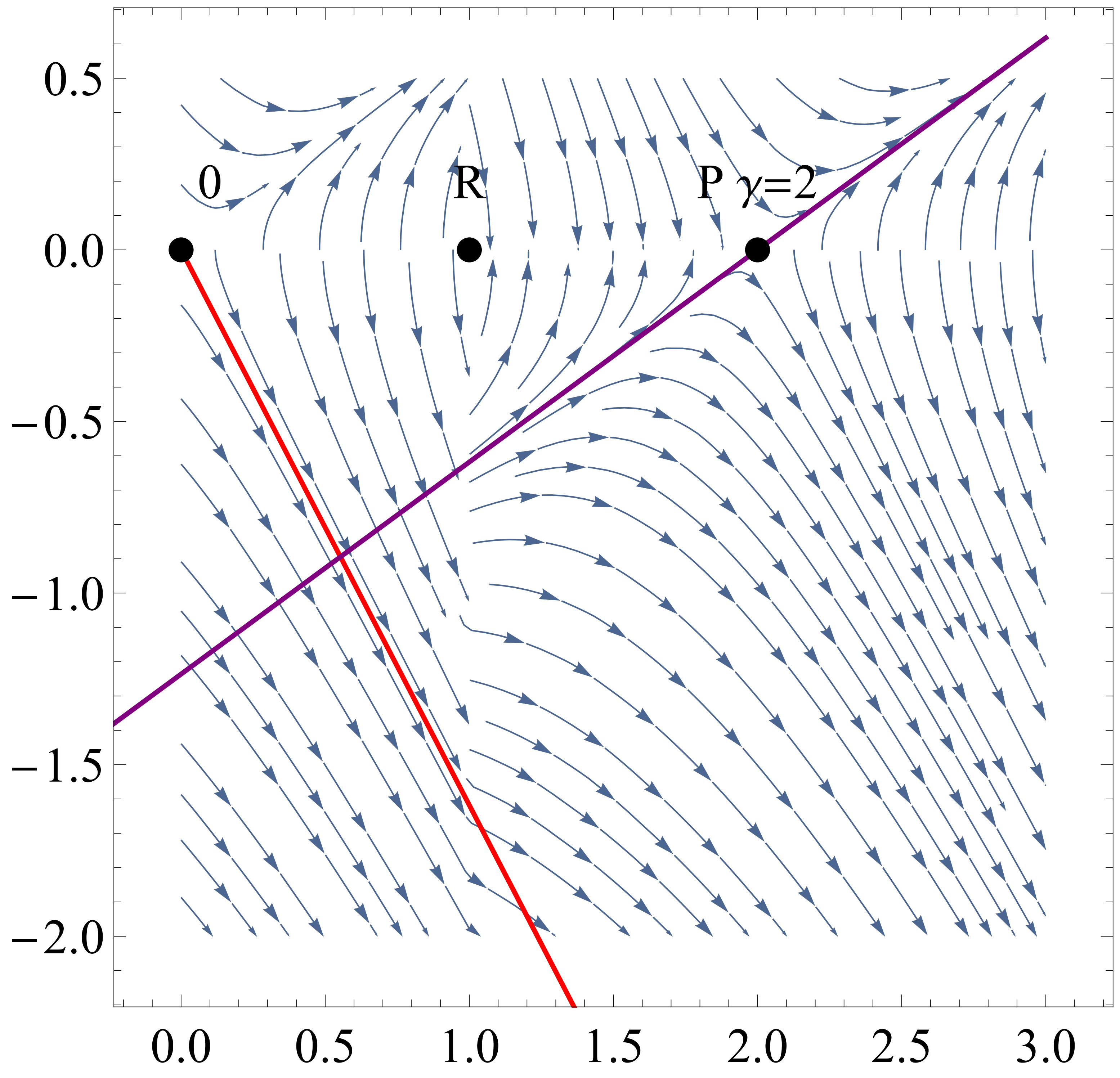
$$\frac{\mathrm{d}u}{\mathrm{d}C} = -\alpha - \frac{C}{u}$$

(with $D = 1$) whose solution satisfying $u(0) = 0$ is

$$u = -\kappa_+ C$$

where κ_+ is the positive root of $\kappa^2 - \alpha\kappa = 1$.

Example: determination of the propagation velocity



When $\gamma > C > 1$, then the principal differential equation is

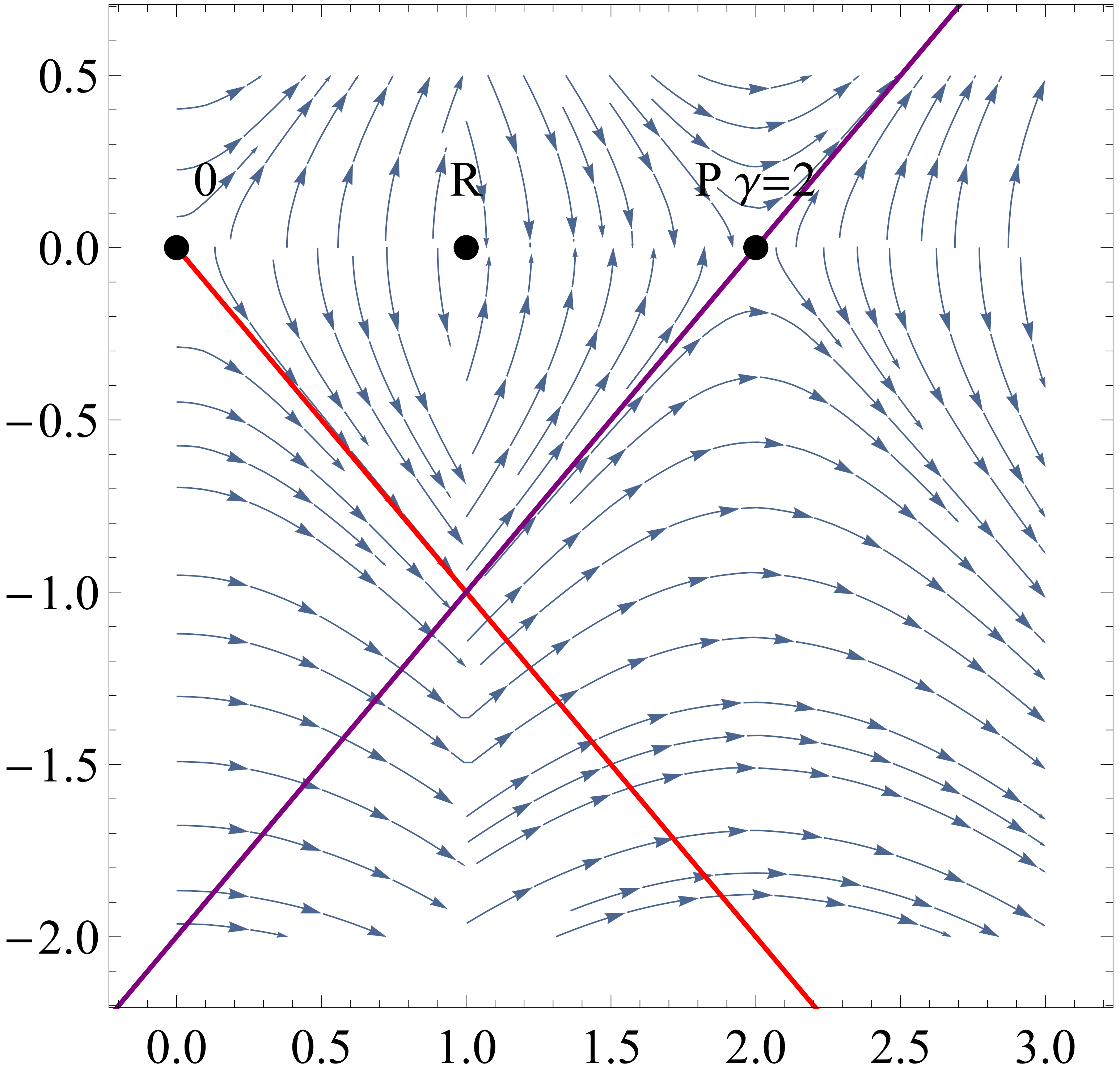
$$\frac{\mathrm{d}u}{\mathrm{d}C} = -\alpha - \frac{\gamma - C}{u}$$

whose solution satisfying $u(\gamma) = 0$ is

$$u = -\kappa_{-}(C - \gamma)$$

Problem? The sepatrix should be unique and go from one singular point to the other. As Q is piecewise continuous, $\dot{u}$ is discontinuous at $C = 1$.

Example: determination of the propagation velocity



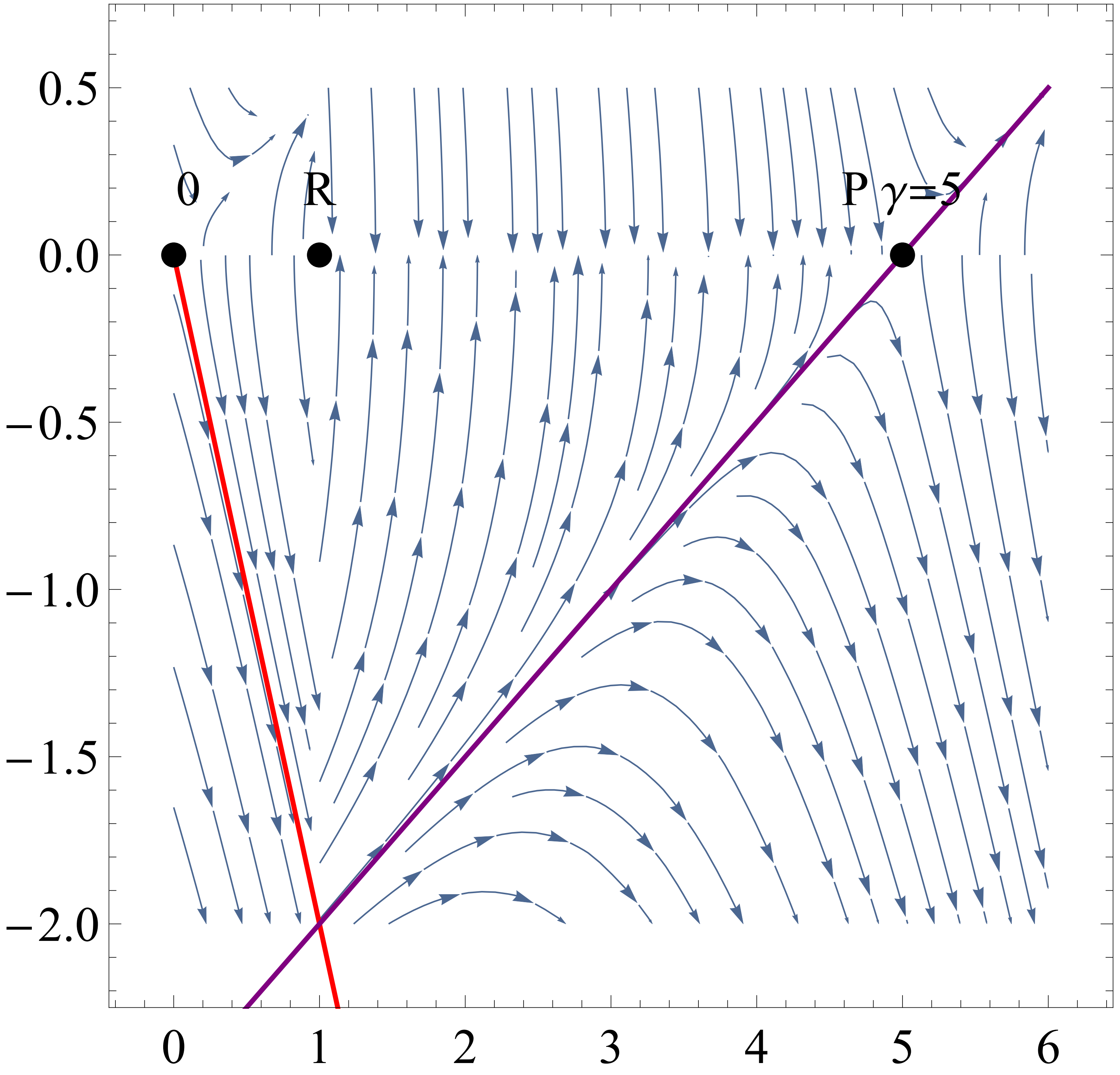
Phase portrait for $\gamma = 2$, so for $\alpha = 0$

The only possibility is to impose the crossing of the two sepatrices at $C = 1$. Therefore

$$-\kappa_-(1-\gamma) = -\kappa_+ \Rightarrow \alpha = \frac{\gamma - 2}{\sqrt{\gamma - 1}}$$

For $\gamma = 2$, $\alpha = 0$ (no motion)

Example: determination of the propagation velocity



Phase portrait for $\gamma = 5$, so for $\alpha = \frac{3}{2}$

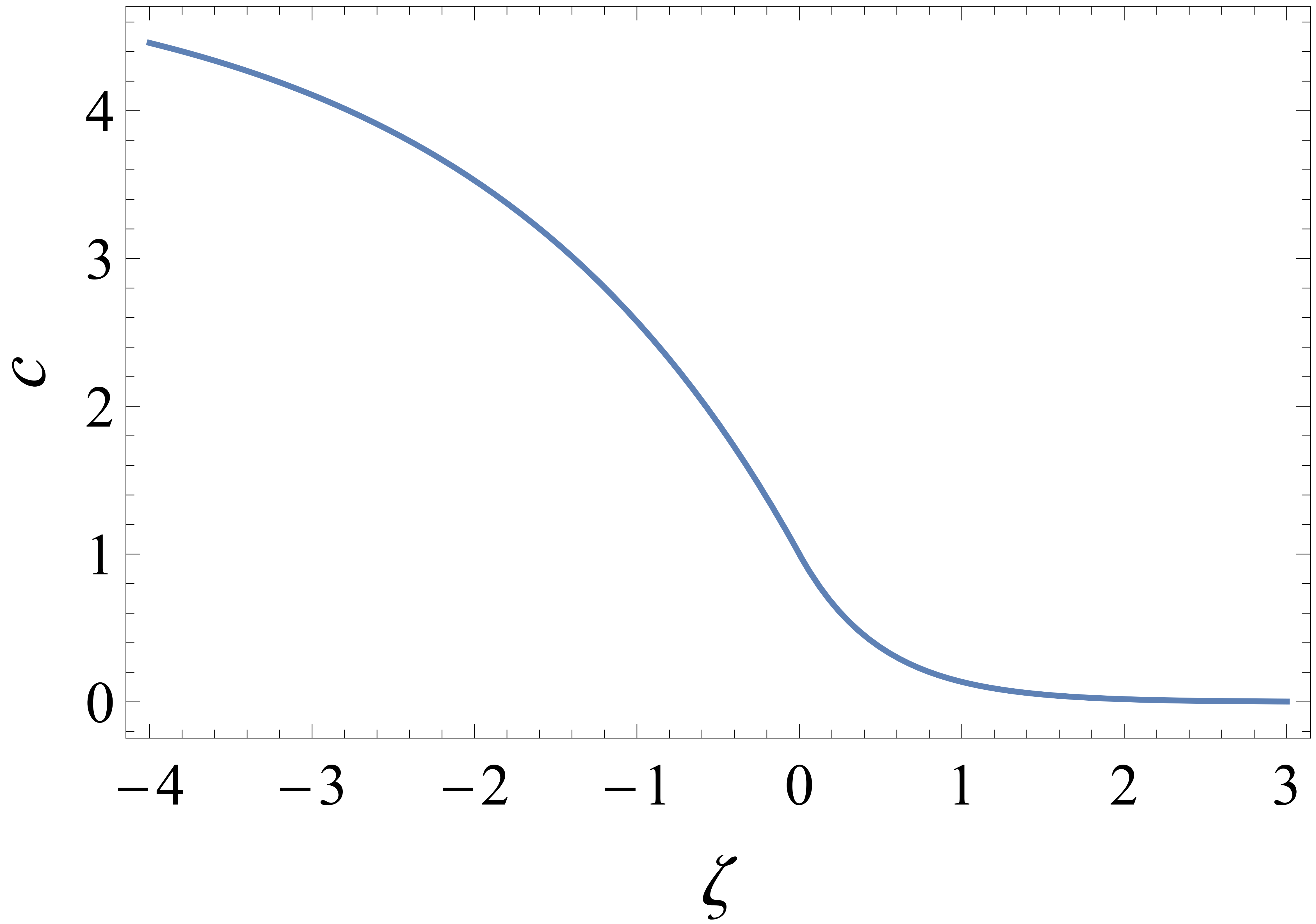
The solution is obtained by integrating $\dot{c} = -\kappa_+ u$ for $c < 1$ and $\dot{c} = -\kappa_-(c - \gamma)$ for $c < 1$. We find

$$c = 5 + \exp(-t/2)a \text{ for } c > 1$$

$$c = \exp(-2t)b \text{ for } c > 1$$

where a and b are two constants of integration. Assuming that $c(0) = 1$, then $a = 4$ and $b = 1$.

Example: determination of the propagation velocity



The solution connects two steady states located at $\pm\infty$ (corresponding to $Q(c) = 0$.

Note the absence of propagation front.

Profile $c(\zeta)$ with $\zeta = x - \alpha t$ for $\gamma = 5$, so for $\alpha = \frac{3}{2}$