Chapter 9: Parabolic problems

Similarity and Transport Phenomena in Fluid Dynamics

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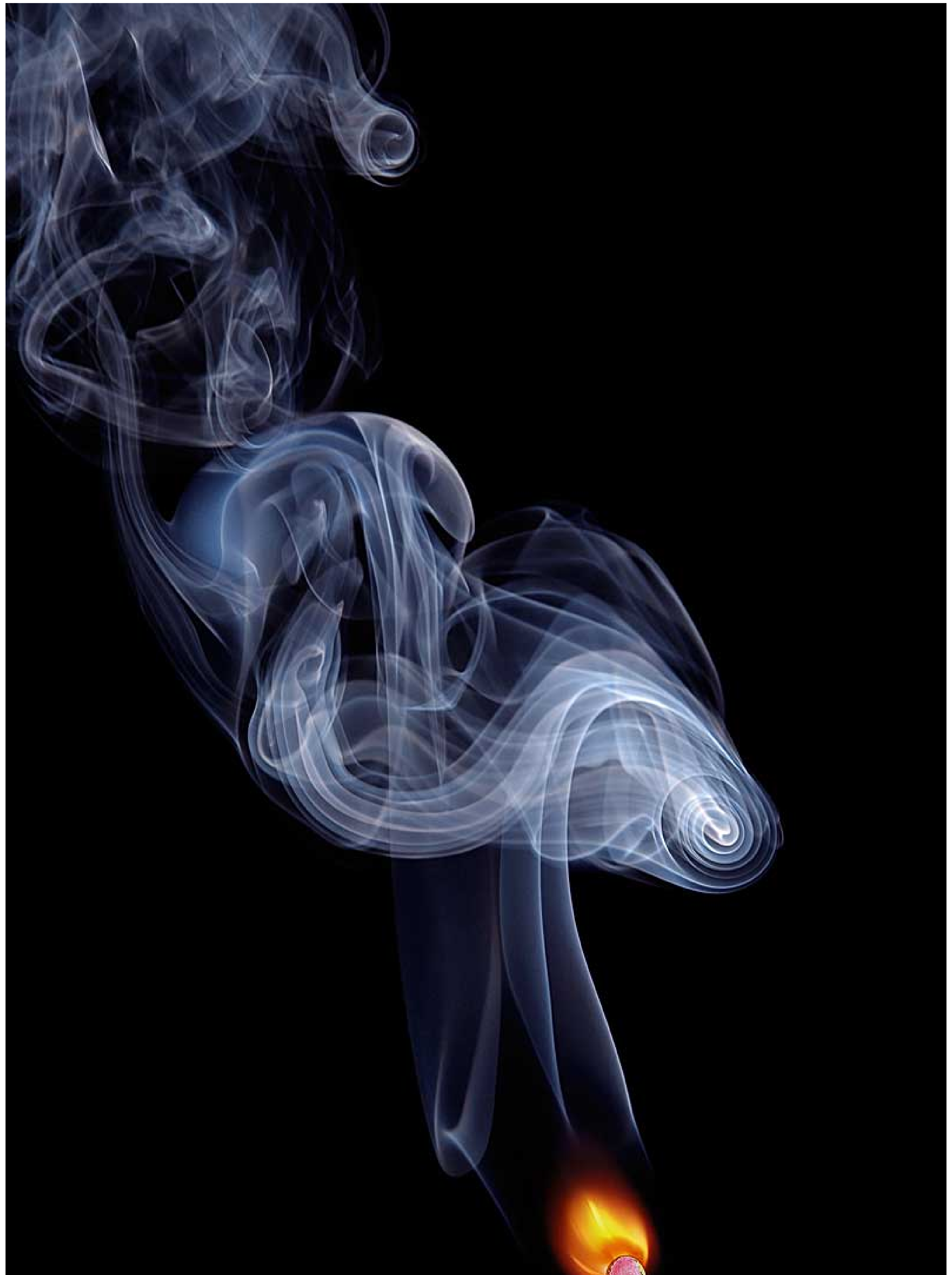


- Linear diffusion
- Nonlinear diffusion
- Boundary-layer problem
- Stefan problem

Linear diffusion

Let us consider the diffusion equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

with $D = 1$ for the sake of simplicity. It is invariant to stretching group

$$c' = \lambda^\alpha, t' = \lambda^2 t, \text{ and } x' = \lambda x$$

for any α value (its value is determined by the boundary and initial conditions). The characteristic equations are

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{2t} = \frac{\mathrm{d}c}{\alpha c}$$

which have the two independent integrals $\xi = x/\sqrt{t}$ and $c/t^{\alpha/2}$, and so the most general solution is

$$c = t^{\alpha/2} C(\xi)$$

Linear diffusion

The ODE associated with the linear diffusion equation is thus

$$\ddot{C} = \frac{\alpha}{2}C - \frac{1}{2}\xi\dot{C}$$

To solve this ODE, we have to provide different boundary initial value problems (BIVP) associated with different α values.

Example: bar with a constant supply at $x = 0$

Consider the BIVP

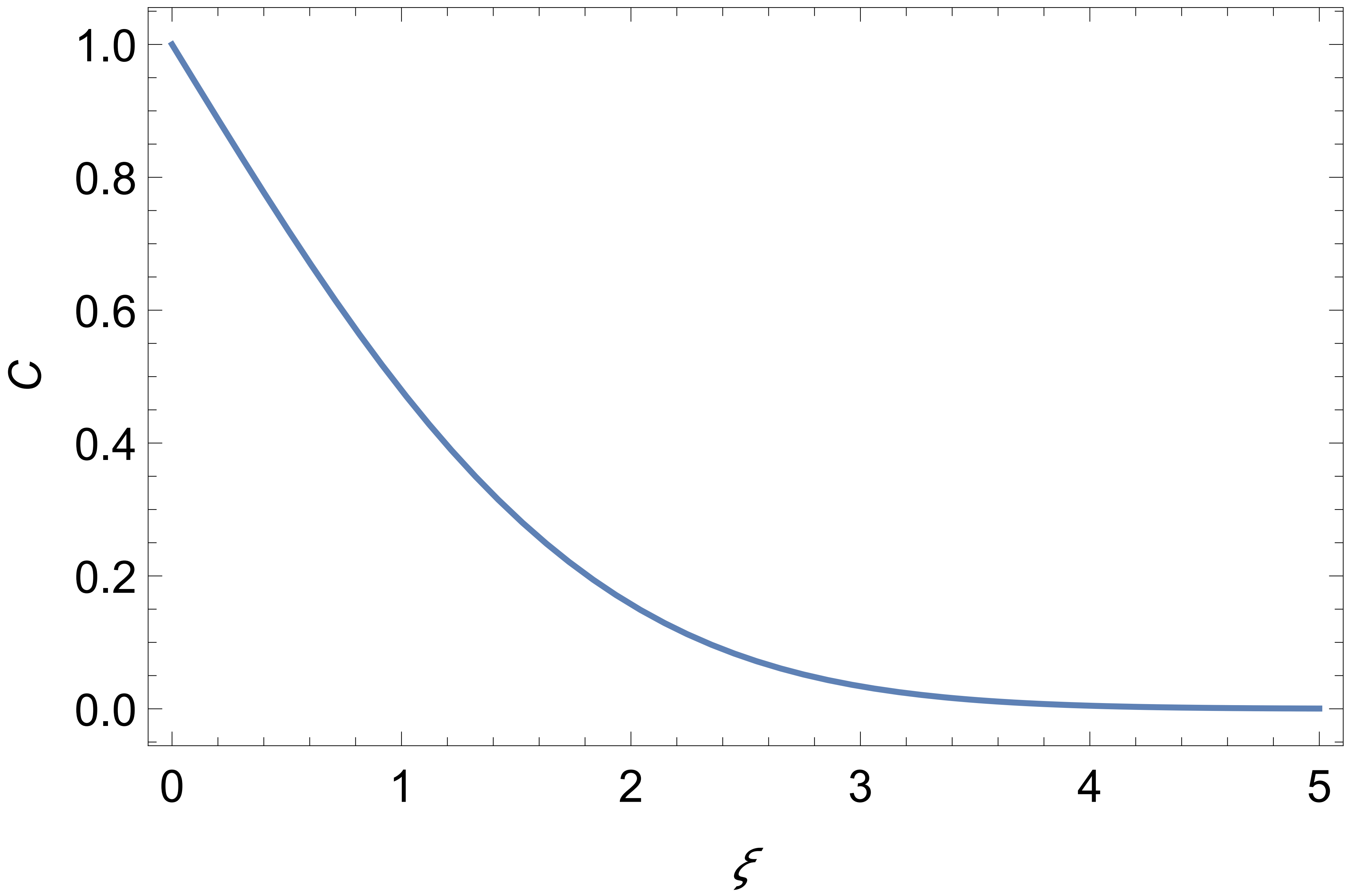
$$c(0,t) = 1 \text{ and } c(\infty,t) = 0$$

for $t > 0$ and $c(x,0) = 0$ for $x > 0$. So this imposes $\alpha = 0$, $C(0) = 1$ and $C(\infty) = 0$.

The solution to the principal ODE is then

$$C = \mathrm{erfc}\left(\frac{\xi}{2}\right)$$

Linear diffusion



Solution to the BIVP

There is no front. Concentration at infinity grows instantaneously (but infinitesimally)

The flux at the left boundary is

$$\phi = \left.\frac{\partial c}{\partial x}\right|_{x=0} = -\frac{1}{\sqrt{\pi t}} \propto t^{-1/2}$$

Nonlinear diffusion

Let us consider this nonlinear diffusion problem with a diffusion coefficient that is a linear function of concentration

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}\left(c\frac{\partial}{\partial t}\right)$$

The equation is invariant to the stretching group

$$c' = \lambda^{\alpha}, t' = \lambda^{\beta} t, \text{ and } x' = \lambda x$$

with $\alpha + \beta = 2$. Two invariants are $c/t^{\alpha/\beta}$ and $\xi = x/t^{1/\beta}$ and the most general solution takes the form

$$c = t^{\alpha/\beta} C(\xi)$$

Nonlinear diffusion: examples

Example: spreading of an initial pulse

Let us consider that initially all the matter is included in a pulse (Dirac). Mass conservation implies

$$\int_{-\infty}^{\infty} c(x,t)\mathrm{d}x = 1$$

And thus $\alpha = -1$ and $\beta = 3$. Concentration is zero at $\pm\infty$: $c(\infty, t) = 0$ and initially, it is also zero: $c(x, 0) = 0$. The most general solution takes the form

$$c = t^{-1/3}C(\xi) \text{ with } \xi = \frac{x}{t^{1/3}}$$

The principal equation is

$$3(\dot{C}^2 + C\ddot{C}) + C + \xi\dot{C} = 0$$

Nonlinear diffusion: examples

Integrating this ODE once gives

$$3C\dot{C} + \xi C = a$$

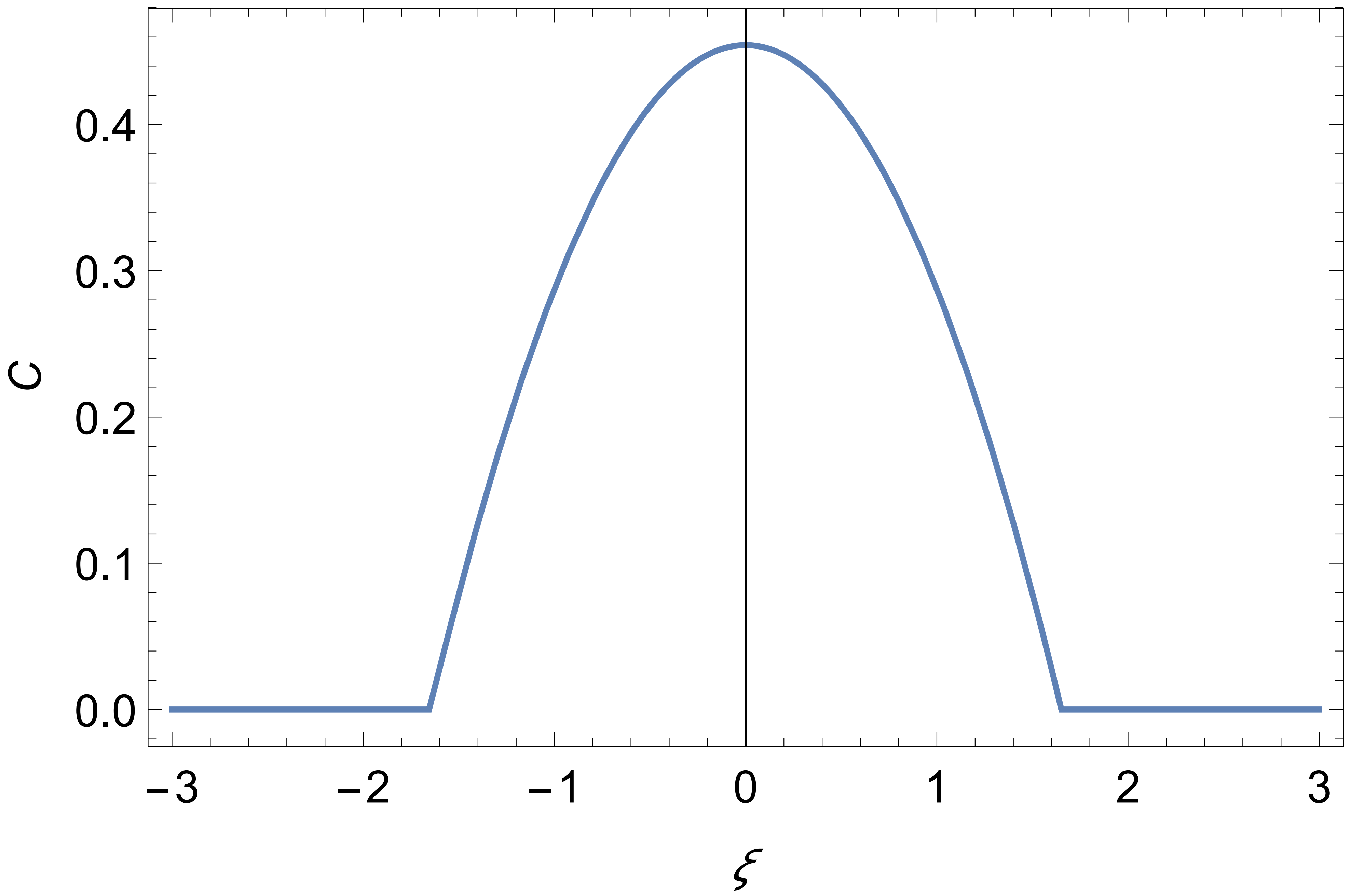
where a is a constant of integration. As the solution is expected to be symmetric ($C(-\xi) = C(\xi)$), then $a = 0$ so that $\partial_x c = 0$ at $x = 0$. Upon integration again, we find

$$C = \frac{b - \xi^2}{6} \text{ for } |\xi| < b$$

where b is constant of integration. For $|\xi| \geq b$, $C = 0$. For Mass conservation implies

$$\int_{-\infty}^{\infty} c(x,t)\mathrm{d}x = 1 = \int_{-b}^{b} C(\xi)\mathrm{d}\xi = \frac{2}{9}b^3 \Rightarrow b = \sqrt[3]{\frac{9}{2}}$$

Nonlinear diffusion: examples



Solution to the BIVP

There is a front at $\xi = \pm b$.

The fronts move at a constant velocity, but declining velocity

$$|x_f| = bt^{1/3} \Rightarrow \dot{x}_f = \pm\frac{b}{3}t^{-2/3}$$

Flux is discontinuous at the front x_f

$$\phi = \left.\frac{\partial c}{\partial x}\right|_{x=x_f} = -t^{-1/3}\frac{b}{3} \propto t^{-1/3}$$

Nonlinear diffusion: examples

Example: constant flux at $x = 0$

Let us consider a more complicated case in which the flux at the origin is constant

$$c\frac{\partial c}{\partial x}\bigg|_{x=0} = -b$$

where b is now a flux constant. The PDE is invariant to the stretching group $\alpha = 1/2$ and $\beta = 3/2$. Concentration is zero at $\pm\infty$: $c(\infty, t) = 0$ and initially, it is also zero: $c(x, 0) = 0$. The most general solution takes the form

$$c = t^{1/3}C(\xi) \text{ with } \xi = \frac{x}{t^{2/3}}$$

The principal equation is

$$3(\dot{C}^2 + C\ddot{C}) - C + 2\xi\dot{C} = 0$$

with $C\dot{C} = -b$ at $\xi = 0$ and $C(\infty) = 0$.

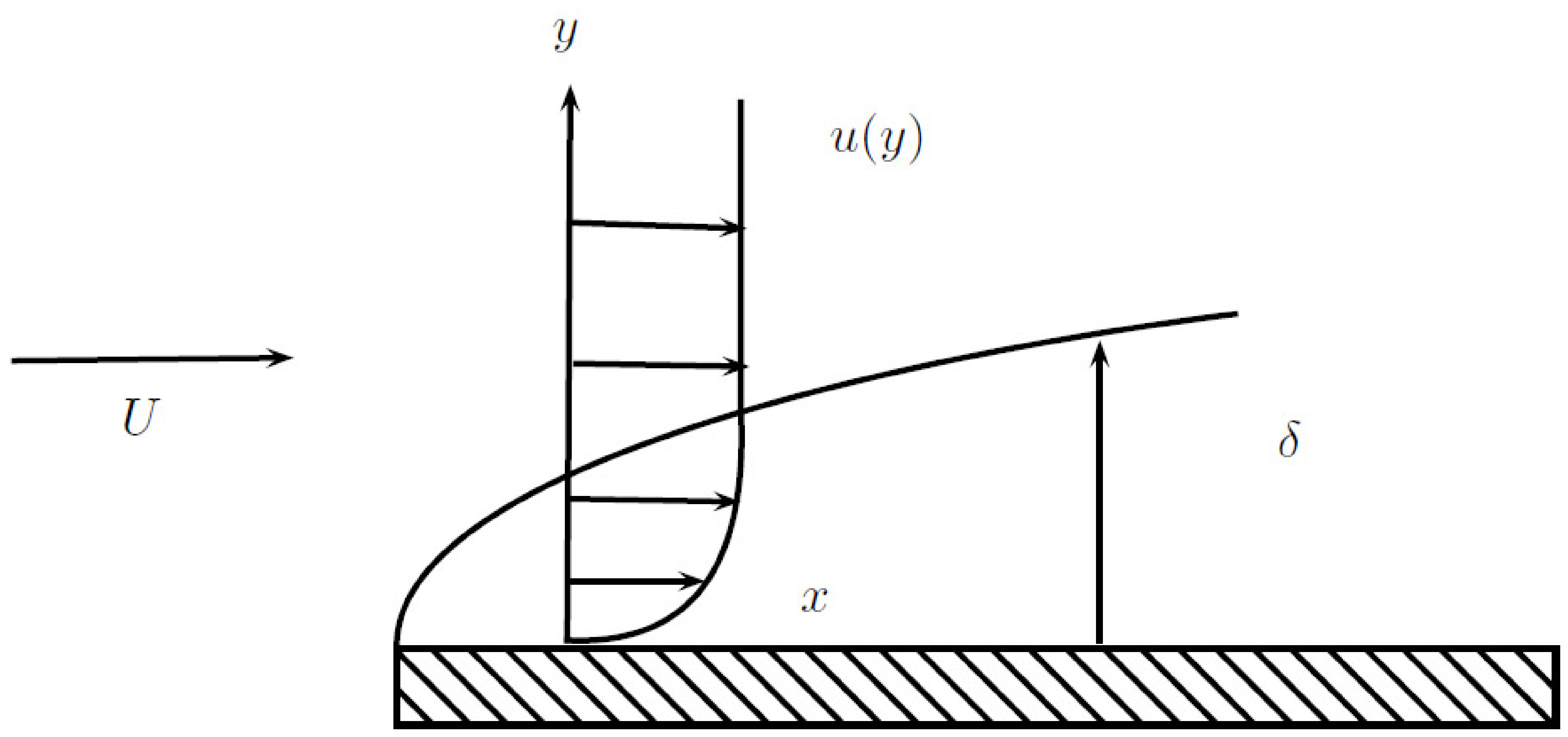
Nonlinear diffusion: examples

The principal equation is not integrable, but it is invariant to the associated group (see chap. 6) $C' = \mu^2 C$ and $\xi' = \mu\xi$. The first invariant is $u = C/\xi^2$ and the first differential invariant is $v = \dot{C}/\xi$. We can thus reduce the order of the principal equation, which becomes

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{u - 2v - 3v^2 - 3uv}{3u(v - 2u)}$$

Numerical solutions can then be sought... see project!

Boundary layer



Boundary layer along a flat plate

For stationary isochoric flows in the vicinity of a solid boundary, scale analysis shows that the Navier-Stokes equation reduce to the Prandl equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(additional assumptions, the incident flow far from the boundary is uniform and there is no pressure gradient), with $\nu = \mu/\varrho$ the dynamic viscosity.

Boundary layer

The boundary conditions are

$$u = v = 0 \text{ at } y = 0 \text{ for } x > 0$$

$$u = U \text{ when } y \to \infty$$

The Prandl equations are invariant to the stretching group

$$x' = \lambda^2 x, y' = \lambda y, v' = \lambda^{-1} v, \text{ and } u' = u$$

A convenient way of solving the Prandl equations is to use the stream function

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

The similarity form is $\psi = \sqrt{x} f(\xi)$ with $\xi = y/\sqrt{x}$.

Boundary layer

To make the problem dimensionless, we pose

$$\psi = \sqrt{\nu x U}\, f(\xi) \text{ with } \xi = \frac{y}{\sqrt{xU}}$$

The PDE is equivalent to the third-order differential equation (called the Blasius equation)

$$2f^{(3)} + f\ddot{f} = 0$$

subject to $\dot{f}(0) = 0$, $f(0) = 0$, and $\dot{f}(\infty) = 1$. The Blasius equation is invariant to the stretching group

$$f' = \lambda^{-1} f \text{ and } \xi' = \lambda \xi$$

Boundary layer

Great! We can reduce the order of the ODE by one, but we then end up with a second-order ODE, and it is not possible to use the phase portrait technique... What can we do?

- still reduce the order of the ODE (see Hydon, CUP, 2000)
- solve the ODE numerically: exact shooting method
- use the von Mises transformation (hodograph transformation)

Shooting method: traditional solvers are unable to solve boundary value problems like the Blasius equation. Instead, we have to provide the input $f(0)$, $\dot{f}(0)$ and $\ddot{f}(0)$. So the idea is to select a trial value for $\ddot{f}(0)$, and increment it until the boundary condition $\dot{f}(\infty) = 1$ is satisfied.

Boundary layer: shooting method

The shooting method can be accelerated by taking similarity into consideration. Indeed, the solution is invariant to the (twice-extended) stretching group

$$\xi' = \lambda\xi, f' = \lambda^{-1}f, \dot{f}' = \lambda^{-2}\dot{f}, f'^{(3)} = \lambda^{-3}f^{(3)}$$

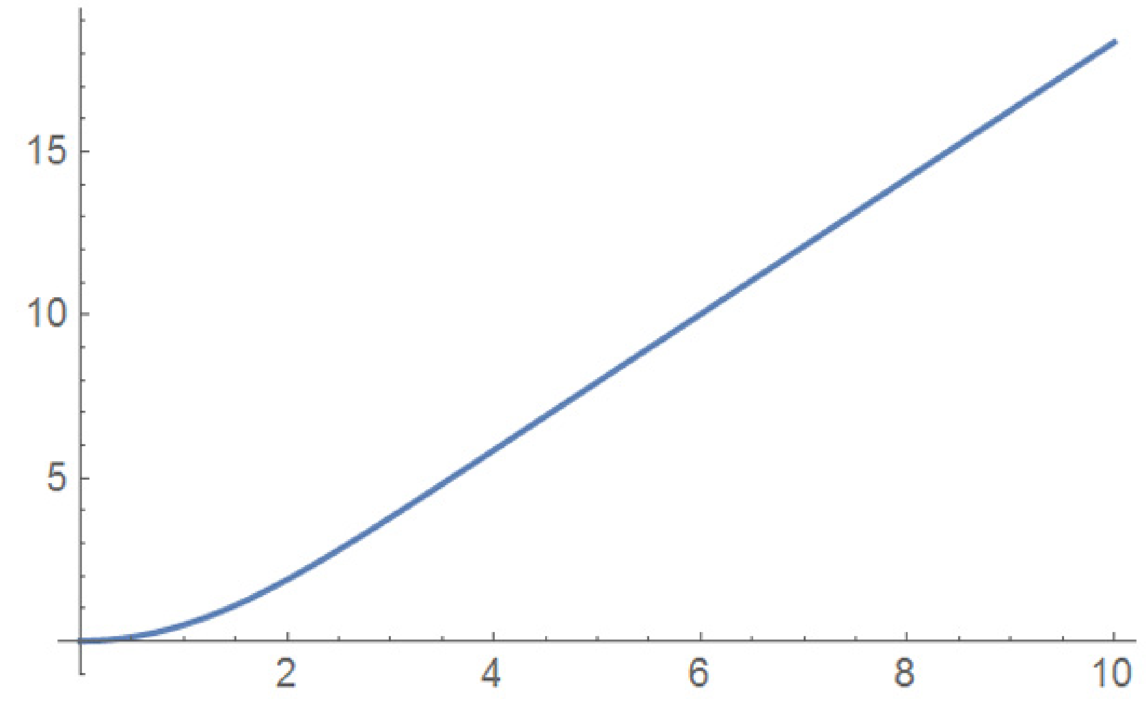
We can use this information to find the right value of $\dot{f}(0)$. Indeed, if we solve the ODE with one trial value for $\ddot{f}(0)$, we can then determine the value of $\dot{f}(\infty)$ (say, $\dot{f}(\infty) = a$). We then seek λ such at $\dot{f}' = \lambda^{-2}\dot{f} = 1$.

So let us try to solve the ODE with $\ddot{f}(0) = 1$.

Boundary layer: shooting method

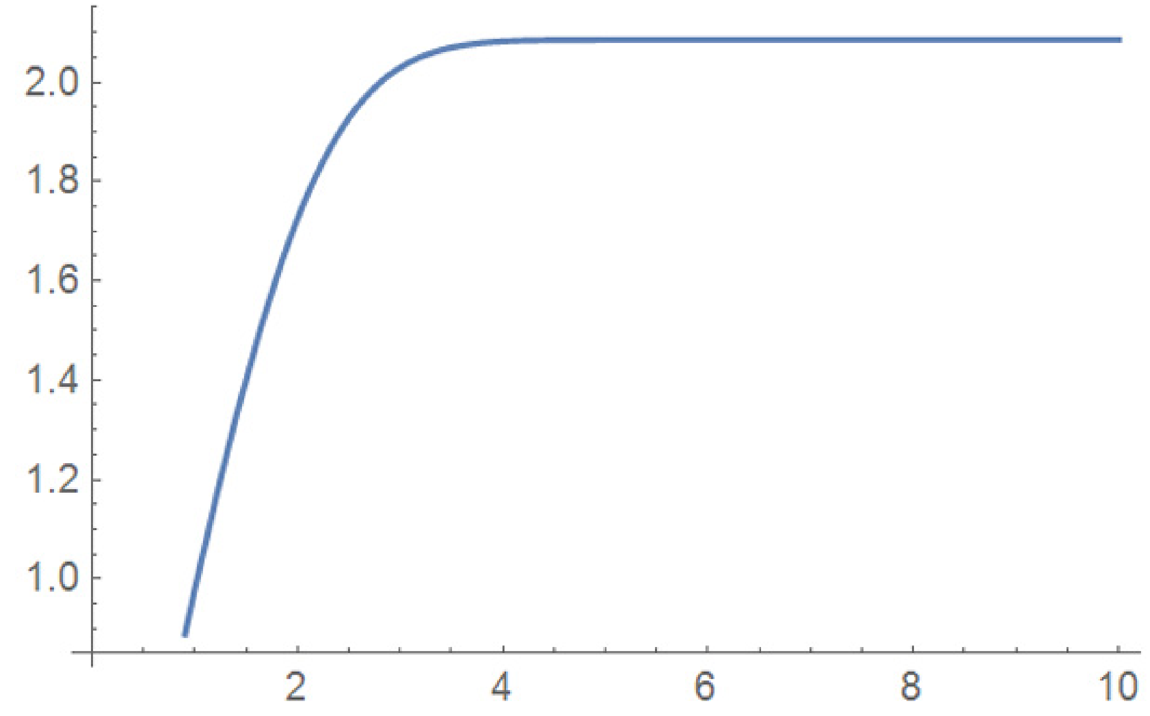
In[13]:= `Plot[f[x] /. eqn, {x, 0, 10}]`

Out[13]=



In[14]:= `Plot[f'[x] /. eqn, {x, 0, 10}]`

Out[14]=



With Mathematica, I seek a numerical solution using NDSolve

eqn = NDSolve[
2 f'''[x] + f[x] f''[x] == 0,
f[0] == 0, f'[0] == 0, f''[0] == 1, f, x, 0,
10]

I then determine λ (called x below)

Solve[(f'[10] /. eqn [[1]])/x^ 2 == 1, x]

I find $\lambda = 1.44409$. So the right value for $\ddot{f}(0)$ is $\ddot{f}(0) = 1/\lambda^3 = 0.332061$.

Boundary layer: von Mises transformation

The idea is to switch the roles played by dependent and independent variables. We treat $\eta = x$ and ψ as the independent variables, whereas y and x are the dependent variables (see Schlichting, H., and K. Gersten, *Boundary Layer Theory*, Springer, Berlin, 2000, pp. 157–158)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \psi}\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial \eta} - v\frac{\partial}{\partial \psi}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial}{\partial \psi}\frac{\partial \psi}{\partial y} = u\frac{\partial}{\partial \psi}$$

and so the momentum balance equation becomes a nonlinear diffusion equation

$$\frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \psi}\left(\nu u \frac{\partial u}{\partial \psi}\right)$$

Boundary layer: von Mises transformation

The PDE is subject to the boundary conditions

$$u(\eta,0)=0 \text{ and } \lim_{\psi\to\infty} u(\eta,\psi)=U$$

The PDE is invariant to the stretching group

$$\psi'=\lambda\psi,\eta'=\lambda^a\eta,u'=\lambda^b u$$

with $a+b=2$, $b=0$, and so $a=2$. The similarity variable is thus $\xi=\psi/\sqrt{\eta}$ and the principal equation is a second-order ODE

$$f\ddot{f}+\dot{f}^2+\frac{1}{2}\xi\dot{f}=0$$

where we set $u(\eta,\psi)=f(\xi)$. (I set ν and U to unity for simplicity).

Boundary layer: von Mises transformation

This ODE is invariant to

$$\xi' = \lambda\xi \text{ and } f' = \lambda^2 f$$

We introduce the invariant $p = f/\xi^2$ and the first differential invariant $q = \dot{f}/\xi$, the ODE can be reduced to a first-order ODE

$$\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{q(2q + 2p + 1)}{2p(2p - q)}$$

Exercise (homework): study the phase portrait and deduce how the solution behaves in the limit $\xi \to 0$. How is this instrumental in determining the wall shear stress?

Stefan problem

The Stefan problem describes how temperature varies in a homogeneous medium undergoing a phase change (water to ice). The problem has one (or two) moving boundary, which is the interface at which the medium passes from one state to another. The main governing equation is the heat equation.
Whereas the equations are linear, the system of equations is nonlinear. In nonlinear parabolic problems, the interface (or front) moves at a constant velocity, which is usually unknown (it is determined in the course of solving the equations).

Stefan problem

The initial boundary value problem to solve includes the (linear) diffusion equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

(diffusivity set to unity), and the Neumann boundary conditions at the left (fixed) boundary due to the inlet heat flux

$$\frac{\partial T}{\partial x}(0, t) = -f(t)$$

the boundary conditions at the moving boundary $x = s(t)$

$$T(s(t), t) = 0 \text{ and } \frac{\partial T}{\partial x}(s, t) = -\dot{s}$$

The initial conditions are $T(x, 0) = 0$ and $s(0) = 0$.

Exercises 1--2: Stefan problem

1. Show that the problem is nonlinear (hint: use the change of variable $\xi = x/s(t)$)

2. When $f = 0$, show that the governing equations are invariant to a stretching group. Determine the solution.

Stefan problem: numerical solution

The problem is the moving boundary. To fix it, we use a change of variable $\xi = x/s(t)$ such that the computational domain is $[0, 1]$. How to solve the equations?

The equation is a nonlinear parabolic equation, we can use pdepe in Matlab.

Another possibility is to use the method of lines. We discretize the diffusion term

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\delta x^2}(T_{i+1} + T_{i-1} - 2T_i) + O(\delta x^2)$$

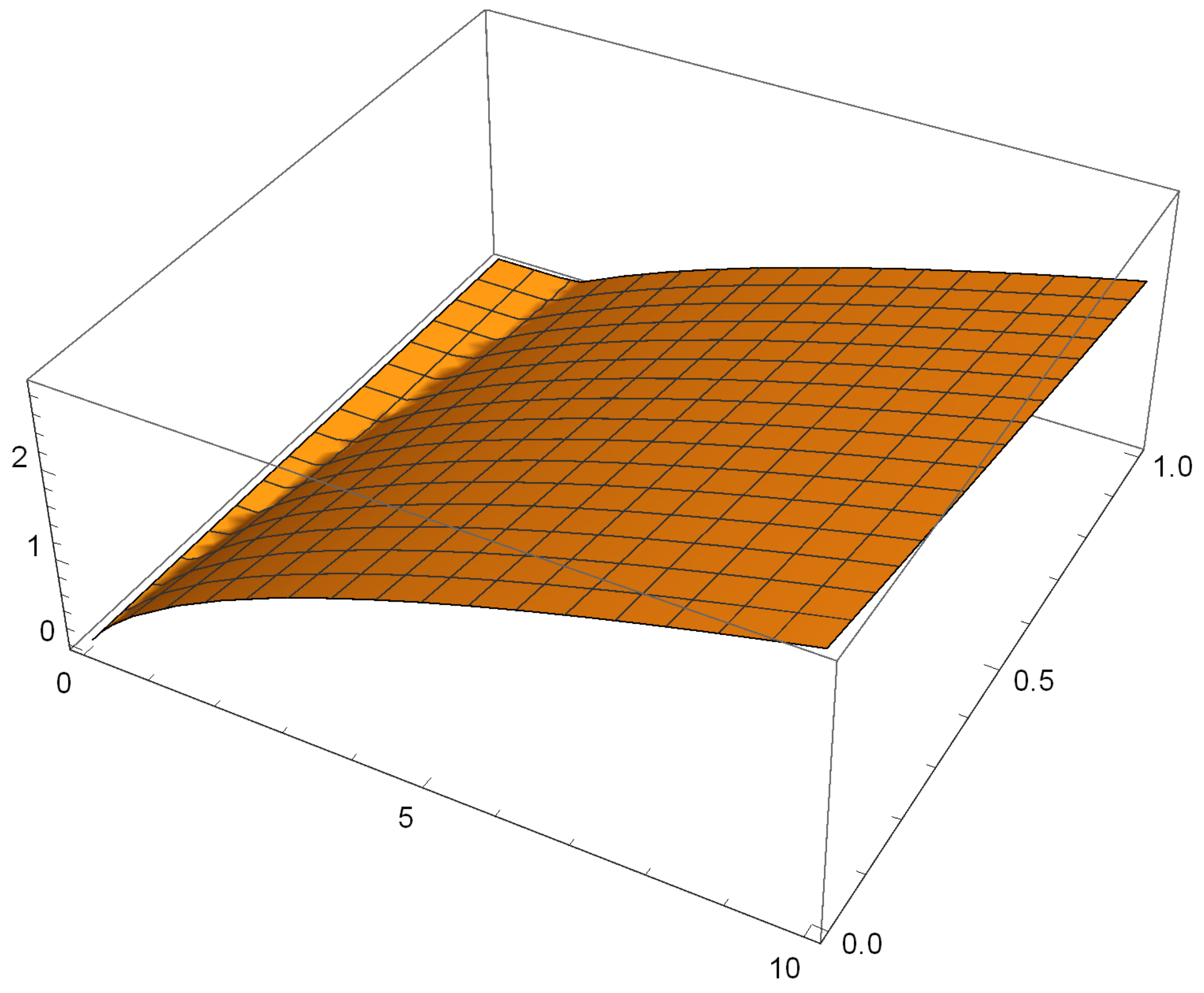
We then solve a system of coupled ODEs

$$\dot{T}_i = \frac{1}{\delta x^2}(T_{i+1} + T_{i-1} - 2T_i)$$

using standard solvers (ode45 in Matlab, NDSolve in Mathematica)

Meyer, G.H., One-dimensional parabolic free boundary problems, *SIAM Rev.*, 19, 17-34, 1977.

Stefan problem: numerical solution



The problem is that the trick does not always perform well. Depending on the boundary conditions imposed, the solution may be singular at short times. A number of techniques have been developed to fix this issue (Marshall, G., A front tracking method for one-dimensional moving boundary problems, *SIAM J. Sci. Statist. Comput.*, 7, 252-263, 1986.) One of them involves finding approximate similarity solutions at short times (Mitchell, S.L., and M. Vynnycky, Finite-difference methods with increased accuracy and correct initialization for one-dimensional Stefan problems, *Appl. Math. Comput.*, 215, 1609-1621, 2009.).

Stefan problem: numerical solution

Without loss of generality we can see solutions in the form

$$T = h(t)F(\xi, t) \text{ with } \xi = \frac{x}{s(t)}$$

The governing equation becomes

$$h\frac{\partial^2 F}{\partial \xi^2} = s\left(s\dot{h}F + sh\frac{\partial F}{\partial t} - \xi\dot{s}h\frac{\partial F}{\partial \xi}\right)$$

subject to

$$F = 0 \text{ and } h\frac{\partial F}{\partial \xi} = -s\dot{s} \text{ at } \xi = 1$$

and

$$h\frac{\partial F}{\partial \xi} = -sf \text{ at } \xi = 0$$

Stefan problem: numerical solution

In the limit $t \to 0$, the governing PDE can be simplified, which makes it possible to work out similarity solutions. This solution can then be used as the initial condition for the numerical scheme.

Exercise 3. Assume that $f(t) = t^{\alpha}$ with $\alpha > 0$. Find an approximate similarity solution that holds at short time.