5.2

$$4\ddot{y} + 9x\dot{y}^{5/3} = 0 \qquad (1)$$

IC $\begin{cases} y(0) = cst \\ y(\infty) = 0 \end{cases}$

a) Invariance?

- translation $y \to y + \lambda$
- stretching $x \to \lambda x$

$y \to \lambda^{\beta} y$

$$4\lambda^{\beta-2}\ddot{y} + 9\lambda^{1+\frac{5}{3}(\beta-1)} x\dot{y}^{5/3} = 0$$

so $\beta - 2 = 1 + \frac{5}{3}(\beta - 1)$

$$\beta\left(1 - \frac{5}{3}\right) = 3 - \frac{5}{3}$$

$$\boxed{\beta = -2}$$

$\Rightarrow$ There are two groups leaving (1) invariant

- translation $\Gamma_1 = \partial_y$
- stretching $\Gamma_2 = x\partial_x - 2y\partial_y$

b) Solution using Γ_1

invariants given by

$$\frac{dx}{0} = \frac{dy}{1} = \frac{d\dot{y}}{0}$$

Reminder if $\Gamma = \partial_y$

then $\xi = 0$ $\eta = 1$

and so $\eta_1 = \eta_x + \dot{y}\eta_y - \dot{y}(\xi_x + \xi_y \dot{y}) = 0$

the one extended group is $\Gamma^{(1)} = \partial_y$

Invariants

$$\frac{dx}{0} = \frac{dy}{1} = \frac{d\dot{y}}{0} \Rightarrow \text{two invariants}$$

$$\begin{cases} p = x \\ q = \dot{y} \end{cases}$$

we make the change of variables

$$p = x \,,\; q = \dot{y}$$

$$(1) \Longleftrightarrow \quad \frac{dq}{dp} = -\frac{9}{4} p q^{5/3}$$

$$\Longleftrightarrow \quad p\,dp + \frac{4}{9} \frac{dq}{q^{5/3}} = 0$$

$$\Longleftrightarrow \quad \frac{1}{2}p^2 - \frac{2}{3} q^{-2/3} = a'$$

(a', a constant of integration)

$$q = \left(\frac{3}{4}p^2 + a\right)^{-3/2} \qquad (2)$$

$$(2) \Rightarrow \quad \dot{y} = -\frac{1}{\left(\frac{3}{4}x^2 + a\right)^{3/2}}$$

$$y = b - \frac{2x}{a\sqrt{4a + 3x^2}}$$

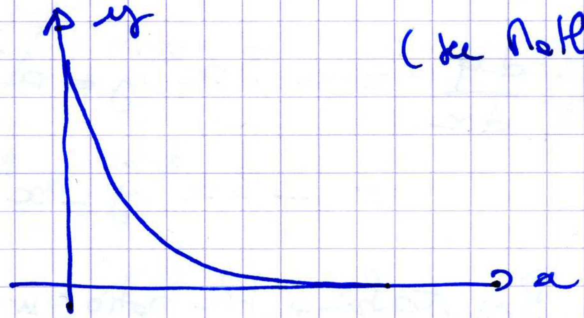
$\underline{IC}$: $y(\infty) = 0$

$$-b + \frac{2}{\sqrt{3}a} = 0 \Rightarrow b = +\frac{2}{\sqrt{3}a},$$

$$y = \frac{2}{a}\left(+\frac{1}{\sqrt{3}} - \frac{x}{\sqrt{4a+3x^2}}\right)$$

$\boxed{NB}$ $$\frac{2x}{a\sqrt{4a+3x^2}} = \frac{2}{\sqrt{3}a} - \frac{4}{3\sqrt{3}}\frac{1}{x^2} + \mathcal{O}\left(\frac{1}{x^4}\right)$$

Plot (see Mathematica notebook)



c) Solution using Γ_2

invariants given by

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{d\dot{y}}{-3\dot{y}}$$

$\underline{\text{Reminder}}$ if $\Gamma = x\partial_x - 2y\partial_y$

then $\xi = x$ $\eta = -2y$ $\eta_1 = -3\dot{y}$

$\underline{\text{Invariants}}$

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{d\dot{y}}{-3\dot{y}}$$

Note the $\frac{5}{3}$ power in (1). As we expect that $y > 0$ and $y \to 0$ (decreasing: $\dot{y} < 0$)

we define the invariants

$$p = x^2 y$$
$$q = x^3 \dot{y}$$

$$\frac{dp}{dx} = 2xy + x^2\dot{y} = (2p+q)\frac{1}{x}$$

$$\frac{dq}{dx} = 3x^2\dot{y} + x^3\ddot{y}$$

$$= \frac{1}{x}\left(3q - x^4 \frac{9}{4} x \dot{y}^{5/3}\right)$$

$$= \frac{1}{x}\left(3q - \frac{9}{4} q^{5/3}\right)$$

$$= \frac{1}{x} q \left(3 - \frac{9}{4} q^{\frac{2}{3}}\right)$$

We take the ratio:

$$\frac{dq}{dp} = q\,\frac{3 - \frac{9}{4}|q|^{\frac{2}{3}}}{2p+q} \qquad (3)$$

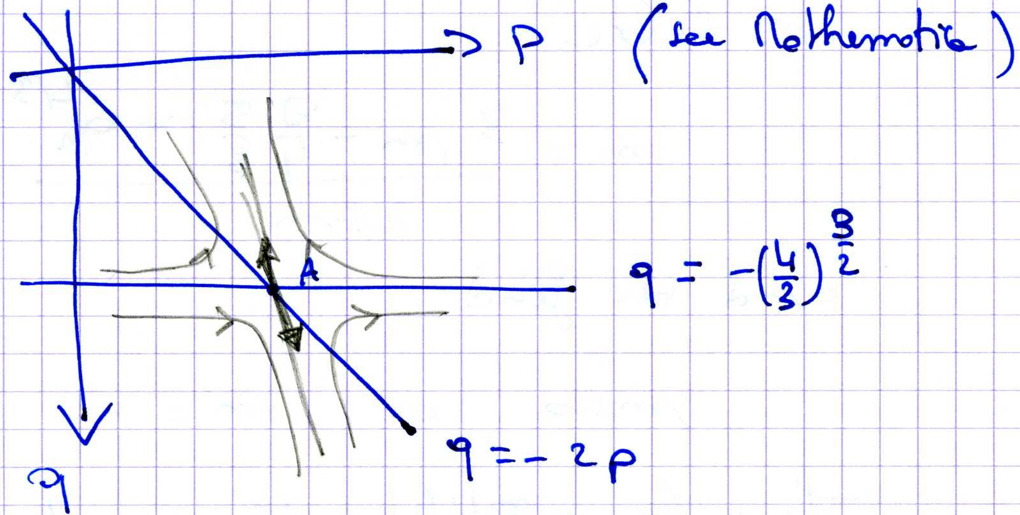
Asymptotic solution

we seek $y = Ax^{+\beta} = Ax^{-2}$

$$4(-2)(-3)Ax^{-4} = -9x\left(-2Ax^{-3}\right)^{5/3}$$

$$24A = -9(-2A)^{5/3}$$

$$2^5 A^2 = +\left(\frac{8}{3}\right)^3 \Rightarrow A = \frac{4}{3\sqrt{3}}$$



A: asymptotic point (saddle)
O: origin (node)

The solution is the separatrix.
To integrate numerically (3), we need to start from the vicinity of

$$q_A = -2p_A \,,\; p_A = A = 4/3/\sqrt{3}$$

because A is a saddle, and there is a single possible path through it.
To find the tangent of the separatrix at A, we use L'hopital's rule

$$m = \lim_{p \to p_A} q \, \frac{3 - \frac{9}{4} q^{2/3}}{2p+q}$$

$$q = q_A + m(p - p_A)$$

Hence $m = \lim_{p \to p_A} \frac{f}{g} = \frac{f'(p_A)}{g'(p_A)}$

We solve

$$m = \frac{3m - \frac{3}{4}\frac{5}{3} m\, q_A^{2/3}}{2+m}$$

and we find

$$m = 0 \quad \text{or} \quad m = -4$$

The only possibility is $m = -4$.

We deduce that the tangent of the separatrix at A is $q = q_A + m(p - p_A)$ with $m = -4$

Numerically, we solve (3) from $p_0 = p_A - \varepsilon$ to $p = 0$. The initial condition is

$$q(p_0) = q_A + m\varepsilon$$

See Mathematica. We obtain a function (interpolating polynomial)

$$q = F(p)$$

Now to return to the original variable we have to solve numerically

$$\dot{y} = x^{-3} F(y x^2)$$

with $y(\varepsilon) = 1$ as initial condition.

Remark 1

Where does Eq. (1) come from?

Some models of superfluid helium assumes that heat transfer is described by the nonlinear Gorter – Mellink law

$$q = -k\left(\frac{\partial T}{\partial z}\right)^{1/3}$$

(When the temperature gradient is negative, then $q = k\left(-\frac{\partial T}{\partial z}\right)^{1/3}$).

This leads to the nonlinear diffusion equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\left(\frac{\partial T}{\partial z}\right)^{1/3}\right)$$

(after appropriate scaling). This equation is invariant to stretching groups. When the boundary condition takes the form $T = T_0$ at $z=0$, the similarity solutions are sought in the form

$$T = y(\xi), \quad \xi = \frac{z}{t^{3/4}}$$

The principal equation is

$$\frac{4}{3}\frac{d}{d\xi}\left(\frac{dy}{d\xi}\right)^{1/3} + \xi\left(\frac{dy}{d\xi}\right) = 0$$

which is equivalent to (1): $4\ddot{y} + 3x\dot{y}^{5/3} = 0$

Remark 2

How to solve (1) numerically?

$$4\ddot{y} + 3x\dot{y}^{5/3} = 0$$
$$y(0) = a > 0$$
$$y(\infty) = 0$$

This is a two-point boundary value problem. The method of exact shooting seen in course does not hold here because the solution is a special solution (separatrix). This is the only integral path issuing from A in the phase portrait (asymptotic behaviour). Here we have to proceed by trial and error by assuming the value of $\dot{y}(0)$, set $y(0) = a$, and solve (1) numerically.