We consider the shallow water equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0 \qquad (2)$$

Subject to the following initial conditions

$$u(x,0) = 0 \quad \forall x \qquad (3)$$

$$h(x,0) = \begin{cases} h_0 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases} \qquad (4)$$

We seek out similarity solutions

$x \to \lambda x$

$t \to \lambda^a t$

$h \to \lambda^c h$

$u = \lambda^d u$

$$(1) \Leftrightarrow \lambda^{c-a}\frac{\partial h}{\partial t} + \lambda^{c+d-1}\frac{\partial hu}{\partial x} = 0$$

$$\Rightarrow c-a = c+d-1 \Rightarrow a+d = 1$$

$$(2) \Rightarrow \lambda^{d-a}\frac{\partial u}{\partial t} + \lambda^{2d-1} u\frac{\partial u}{\partial x} + g\lambda^{c-1} = 0$$

$$d-a = 2d-1 = c-1 \Rightarrow c = 2d$$

$$(4a) \Rightarrow \lambda^c h = h_0 \Rightarrow c = 0$$

Thus $c = d = 0$ and $a = 1$

Similarity forms:

$$\frac{dx}{x} = \frac{dt}{t} = \frac{dh}{0} = \frac{dc}{0}$$

We set $\xi = \frac{x}{t}$

The similarity forms are then

$$u = U(\xi) \quad \text{and} \quad h = H(\xi) \qquad (5)$$

Substituting (5) into (1) and (2) gives

$$\begin{cases} -\dfrac{x}{t^2} H' + \dfrac{1}{\cancel{t}}\left(U'H + H'U\right) = 0 & (6) \\[2ex] -\dfrac{x}{\cancel{t^2}} U' + \dfrac{1}{\cancel{t}} UU' + \dfrac{1}{\cancel{t}}\, g H' = 0 & (7) \end{cases}$$

ξ

We then have the system

$$\begin{cases} HU' + (U - \xi) H' = 0 \\ (U - \xi) U' + g H' = 0 \end{cases}$$

in matrix form

$$\underbrace{\begin{pmatrix} H & U - \xi \\ U - \xi & g \end{pmatrix}} \cdot \begin{pmatrix} U' \\ H' \end{pmatrix} = 0 \qquad (8)$$

$$\underline{\underline{A}} \cdot \underline{X}' = 0 \qquad \text{with } \underline{X} = \begin{pmatrix} U \\ H \end{pmatrix}$$

The system (8) has the trivial solution $U = H = 0$ but this does not satisfy the initial condition (4).
The only possibility is that $\det \underline{\underline{A}} = 0$ (indeterminate system)

$$\det \underline{\underline{A}} \iff gH - (U-\xi)^2 = 0$$

that is,
$$H = \frac{1}{g}(U-\xi)^2 \qquad (9)$$
$$H' = \frac{2}{g}(U'-1)(U-\xi) \qquad (10)$$

Substituting in (8a) or (8b) gives

$$HU' + (U-\xi)H' = 0$$

$$\frac{1}{\cancel{g}}(U-\xi)^2 U' + (U-\xi)\frac{2}{\cancel{g}}(U'-1)(U-\xi) = 0$$

$$(U-\xi)^2(3U'-2) = 0$$

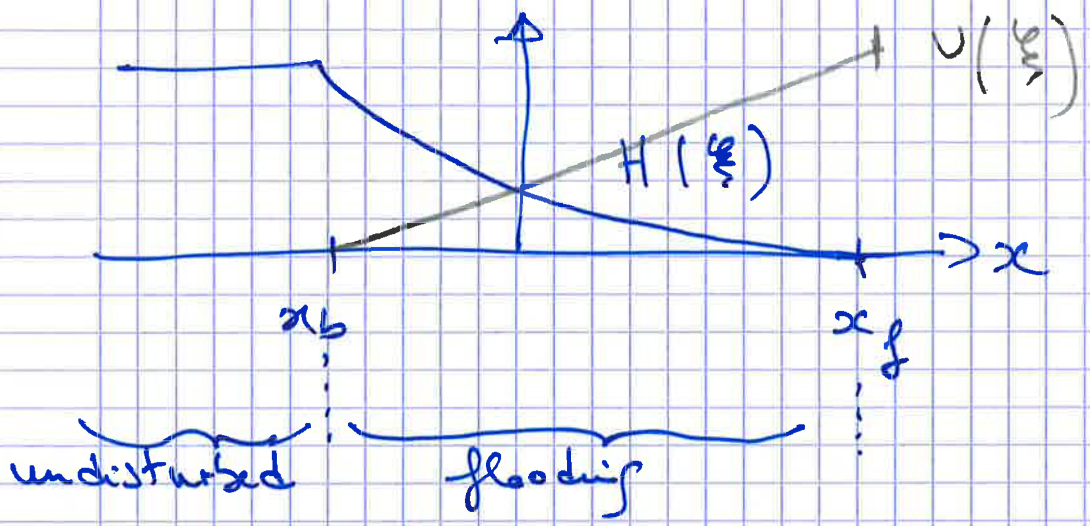
$$\Rightarrow \quad U = \xi \quad \text{or} \quad U' = \frac{2}{3} \qquad (11a)$$
$$U = \frac{2}{3}\xi + c \qquad (11b)$$

with c constant of integration.
(11a) does not satisfy the initial condition.

Combining (11b) and (9) gives

$$H = \frac{1}{g}\left(\frac{2}{3}\xi + c - \xi\right)^2$$

$$= \frac{1}{9g}\left(3c - \xi\right)^2$$



at $\xi = \xi_b$ $\quad U = 0 \quad H = h_0$

$U = 0 \Rightarrow c = -\frac{2}{3}\xi_b$

$H = h_0 \Rightarrow \frac{1}{9g}\left(-2\xi_b - \xi_b\right)^2 = h_0$

$$\boxed{\xi_b = -\sqrt{gh_0}} \qquad (12)$$

at $\xi = \xi_f$ - $\quad H = 0$

$$\frac{1}{9g}\left(+2\sqrt{gh_0} - \xi_f\right)^2 = 0$$

$$\xi_f = 2\sqrt{gh_0}$$

The solutions are then

$$u(x,t) = \frac{2}{3}\frac{x}{t} + \frac{2}{3}c_0 = \frac{2}{3}\left(\frac{x}{t} + c_0\right)$$

$$h(x,t) = \frac{1}{9g}\left(-\frac{x}{t} + 2c_0\right)^2$$

with $c_0 = \sqrt{gh_0}$ celerity of shallow waves

Remarks

1/ parabolic shape of the free surface

2/ front velocity $\dot{\xi}_f = 2c_0$

$x_f = 2c_0 t$

twice the linear wave velocity

3/ at $x=0$ $u(0,t) = \frac{2}{3}c_0$

$h(0,t) = \frac{4}{9}h_0$

Thus the flow rate $q = uh\big|_{x=0} = \frac{8}{27}\sqrt{gh_0^3}$

is constant.