Chap6

exo 5

We consider

$$u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} = \frac{\nu_t}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \qquad (1)$$

BC $\begin{cases} v = 0 & \text{along the } z\text{-axis} \\ v = 0 = u & \text{when } r \to \infty \\ u(0,r) = u_0(r) & z = 0 \end{cases}$

We set $u = \frac{1}{r} \frac{\partial \psi}{\partial r}$ $\quad v = -\frac{1}{r} \frac{\partial \psi}{\partial r}$

$(1) \Leftrightarrow$

$$\psi_r^2 - \psi_r \left(\nu_t - r \psi_{rz} + r \psi_{rr}\right) + r\nu_t\left(\psi_{rr} - r \psi_{rrr}\right) = 0$$

We seek solutions

$$z \to \lambda z$$
$$r \to \lambda^a r$$
$$\psi \to \lambda^b \psi$$

BC $b = a = 1$

$$\frac{dz}{z} = \frac{dr}{r} = \frac{d\psi}{\psi}$$

We set $\psi = z F(\xi)$ with $\xi = \frac{r}{z}$

We find

$$FF' - \xi F'^2 - \xi FF'' + \nu_t\left(-F' + \xi F'' - \xi^2 F'''\right) = 0$$

We divide by ξ^2

$$\nu_t \left(\frac{F''}{\xi} - \frac{F'}{\xi^2} - F'''\right) = \frac{FF''}{\xi} + \frac{F'^2}{\xi} - \frac{FF'}{\xi^2}$$

by integration

$$\nu_t\left(\frac{F'}{\xi} - F''\right) = \frac{FF'}{\xi} + a$$

$\xi = 0 \quad F = F' = 0 \quad \Rightarrow a = 0$

By multiplying by ξ

$$\nu_t\left(F' - \xi F''\right) = FF'$$

$$\nu_t\left(2F - \xi F'\right) = \frac{1}{2}F^2 + a$$

$\xi = 0 \quad F = F' = 0 \quad \Rightarrow a = 0$

$$\frac{\nu\, dF}{2F - \frac{F^2}{2}} = \frac{d\xi}{\xi}$$

$$\Rightarrow \frac{1}{2}\ln\frac{F}{4\nu - F} = \ln\xi + a'$$

$$\boxed{F = \frac{4a\nu\xi^2}{1 + a\xi^2}}$$

$$\Rightarrow u = \frac{1}{r}F' = \frac{1}{r}\frac{8a\xi\nu}{(1 + a\xi^2)^2}$$