Solution to exercise 3 (slide 31)

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Introduction

We study the ordinary differential equation

$$y' = \frac{x + 3xy + 3(1-y)y}{3x(2x+3y)}. \tag{1}$$

We set

$$F = x + 3xy + 3(1-y)y \text{ and } G = 3x(2x+3y), \tag{2}$$

such that the governing equation (1) reads $y' = F/G$. The critical curves are determined by setting $F = 0$ or $G = 0$:

- $F = 0$ leads to
$$x = 3\frac{y(y-1)}{1+3y}. \tag{3}$$
- $G = 0$ leads to $x = 0$ and
$$y = -\frac{2}{3}x \tag{4}$$

Figure 1 shows the phase portrait and the critical curves.

There are two critical points corresponding to $F = G = 0$: point O $(0,0)$ and A $(-3/10, 1/5)$.

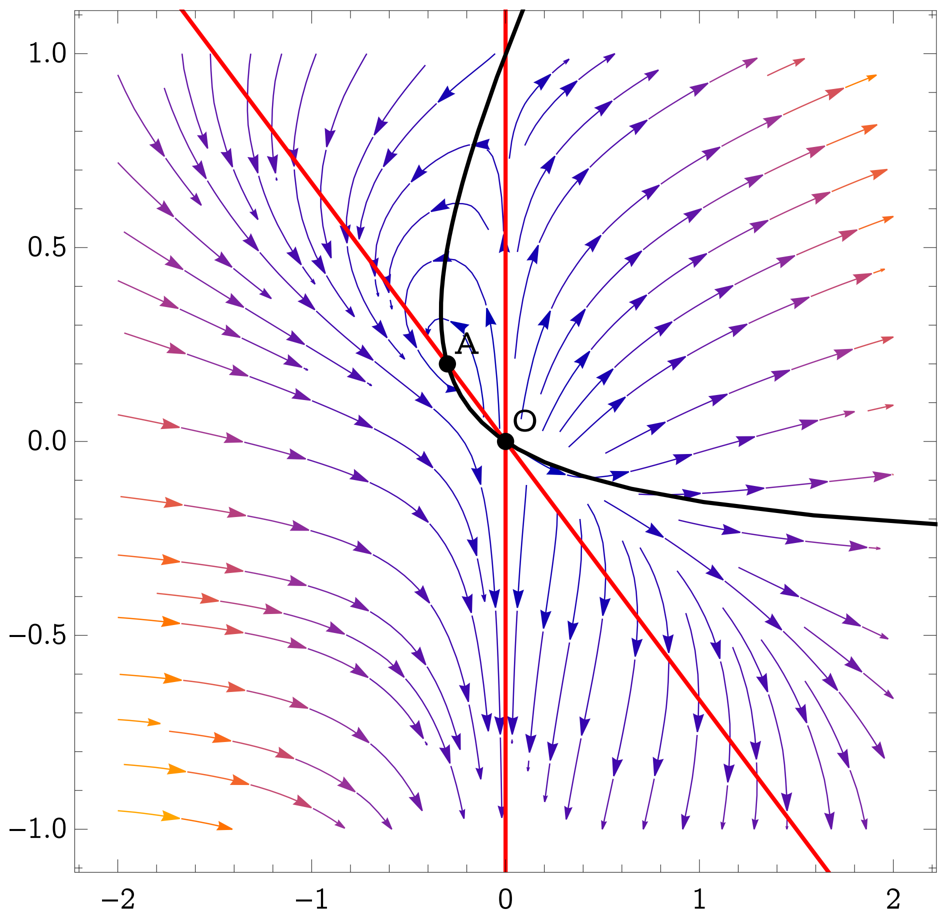

Figure 1: Phase portrait of Eq. (1). The black curve shows $F = 0$ while the red lines show the critical curves $G = 0$.

1 Determination of the solution near O

It is not possible to linearize Eq. (1) at the origin point (multiplicity of the zero root). Point O is a combination of saddle and node points, which is expected because three critical curves cross at this point.

2 Determination of the solution near A

Near point A (-3/10, 1/5), we have

$$F(-3/10 + x, 1/5 + y) = \frac{8}{5}y + \frac{9}{10}y + O(x) + O(y), \tag{5}$$

for $x \to 0$ and $y \to 0$, and

$$G(-3/10 + x, 1/5 + y) = -\frac{9}{5}y - \frac{27}{10}y + O(x) + O(y). \tag{6}$$

If we linearize Eq. (1) around A, we get

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{M} = \begin{pmatrix} -9/5 & -9/5 \\ 8/5 & 9/10 \end{pmatrix} \cdot \boldsymbol{u}. \tag{7}$$

The eigenvalues of $\boldsymbol{M}$ are complex, and thus A is a focal point.

3 Asymptotic behaviour for $x \to \infty$

Let us try to determine the asymptotic behaviour by using the dominant-balance technique when $y \to \infty$ and $x \to \infty$.

Let us first assume that $y \gg x$, then Eq. (1) can be simplified into

$$y' = 3\frac{y^2}{9xy} = \frac{y}{3x} \Rightarrow y = \frac{a}{x^{1/3}} \tag{8}$$

where a is constant of integration. This solution is inconsistent with our initial assumption ($y \gg x$ and $y \to \infty$ when $x \to \infty$). We discard it.

Let us now examine the linear asymptotic solution $y = mx$. Equation (1) can be simplified into

$$m = \frac{3mx^2 + 3(-mx)mx}{3x(2x + 3mx)} = m\frac{1 - m}{2 + 3m} \Rightarrow m = -\frac{1}{4}. \tag{9}$$

Let us eventually assume that $y \ll x$, then Eq. (1) can be simplified into

$$y' = 3\frac{xy}{6x^2} = \frac{y}{2x} \Rightarrow y = a\sqrt{x}, \tag{10}$$

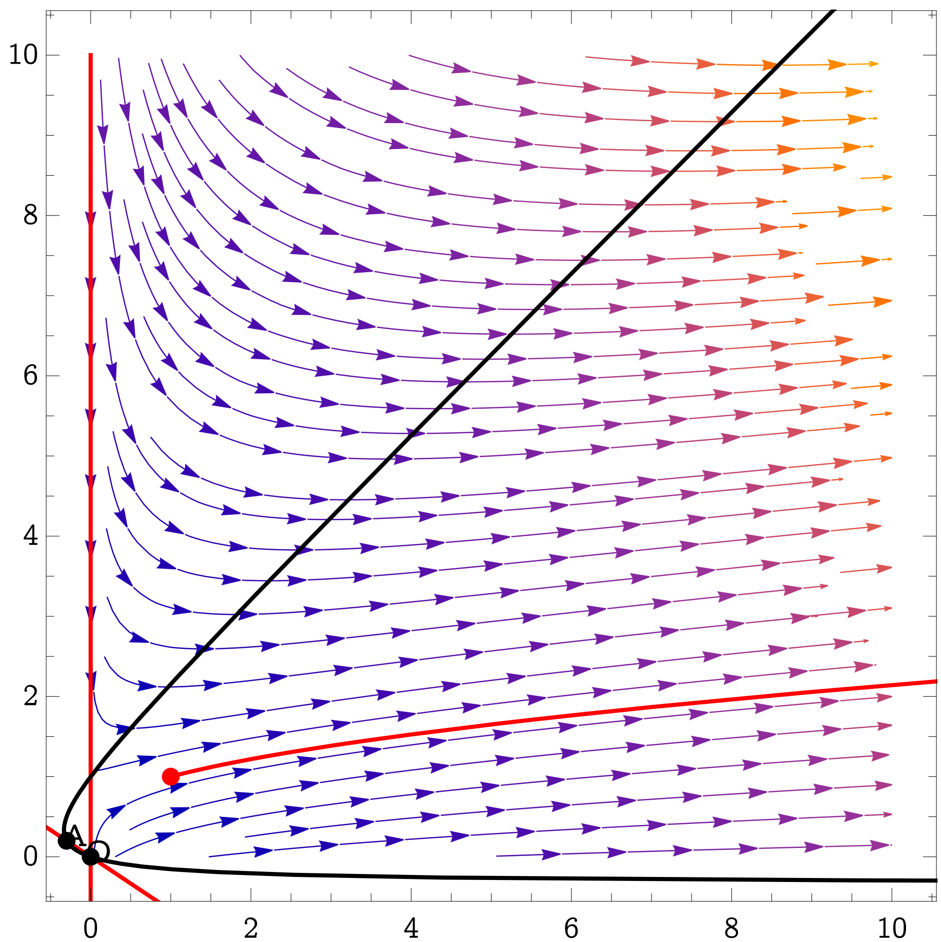

Figure 2: Phase portrait of Eq. (1) and plot of the particular solution to Eq. (1) subject to the initial condition $y(1) = 1$.

where a is constant.

When plotting particular solutions to Eq. (1), it is obvious that they admit asymptotic solutions in the form of Eq. (10), as illustrated by Fig. 2.

The question arises as to whether the asymptotic solution $y = mx$ (for $x \to \infty$) with $m = -1/4$ is a special curve as it does not involve any constant of integration, leading to think that it may be associated with a singular point C (∞, ∞) that is a node or saddle point. To answer this question, we use the following change of variable

$$u = \frac{1}{x} \text{ and } v = \frac{x}{y} \tag{11}$$

such that the singular point C (∞, ∞) in the (x, y) coordinate system becomes B $(0, 1/m)$ in the (u, v) coordinate system. This change of variable can also be written in the following form

$$x = \frac{1}{u} \text{ and } y = \frac{1}{uv}. \tag{12}$$

We can express the governing equation (1) using the new variables (12)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}u} = -u^2\left(-\frac{1}{vu^2} - \frac{1}{uv^2}\frac{\mathrm{d}v}{\mathrm{d}u}\right). \tag{13}$$

We then find that in the system (u, v), Eq. (1) reads

$$v' = \frac{F_{uv}}{G_{uv}} = \frac{-v(-13 + 3(u-1)v + uv^2}{u(-9 + v(u^2 - 6)}. \tag{14}$$

There are thus two critical curves

$$F_{uv} = 0 \Rightarrow u = \frac{3}{v}\frac{4+v}{3+v}, \tag{15}$$

and

$$G_{uv} = 0 \Rightarrow v = \frac{9}{u^2 - 6} \text{ and } u = 0. \tag{16}$$

As expected there is a critical point at B $(0, -4)$ which is the image of the singular point C (∞, ∞). If we linearize Eq. (14) around B, we get

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{M} = \begin{pmatrix} 15 & 0 \\ 16 & -12 \end{pmatrix} \cdot \boldsymbol{u}. \tag{17}$$

where $\boldsymbol{u}$ denotes the vector $(u(t), v(t))$ and the eigenvalues are $\lambda_{\pm} = 15$ and -12 associated with the eigenvectors $\boldsymbol{e}_+ = (27, 16)$ and $\boldsymbol{e}_- = (0, 1)$. The eigenvalues are real and of different sign, the critical point is a saddle point. Figure 3 shows the phase portrait of the governing equation (14).

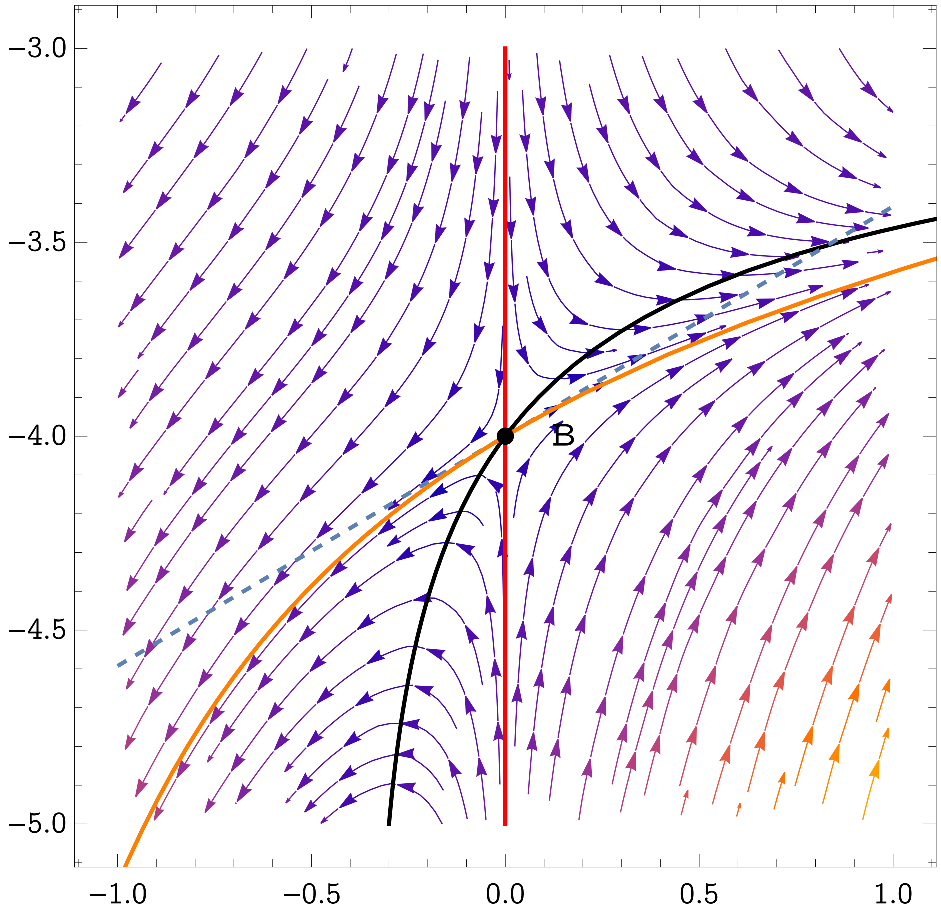

Figure 3: Phase portrait of Eq. (14). The black curve shows $F_{uv} = 0$ while the red lines show the critical curves $G_{uv} = 0$. The dashed line shows the line aligned with the eigenvector $\boldsymbol{e}_+ = (27, 16)$. The orange line shows the numerical solutions to Eqs. (14) and (18).

If we want to find the numerical solution to Eq. (14), that is, the special solution to Eq. (14) passing through point B

$$v'(u) = \frac{F_{uv}}{G_{uv}}, \text{ subject to } v(\epsilon) = -4 + \frac{16}{27}\epsilon \tag{18}$$

for $x \geq \epsilon$ (right branch). To avoid the singularity and the failure of the numerical algorithm, we start from a neighboring point located at $u = \epsilon$ where $\epsilon \ll 1$ is a small number. We do the same for the branch on the left

$$v'(u) = \frac{F_{uv}}{G_{uv}}, \text{ subject to } v(\epsilon) = -4 - \frac{16}{27}\epsilon \tag{19}$$

for $x \leq -\epsilon$. We can then return to the original variables $x = 1/u$ and $y = 1/(uv)$. Figure 4 shows the numerical solution in the (u, v) and (x, y) systems. The numerical result's accuracy depends on the algorithm used. Here we find that $v \to 2.81$ when $u \to \infty$. The left branch ($u < 0$) diverges for $u = -1.93$.

Can we determine the behaviour of the solution near O directly from the governing equation (1)? To do this, we can make an asymptotic expansion of

$$y(x) = \frac{F(x, y(x))}{G(x, y(x))}. \tag{20}$$

To order 1 near $x = 0$, we have

$$f(x) = F(x, y(x)) = f(0) + xf'(0) + O(x^2), \tag{21}$$

where

$$f'(0) = \frac{\partial F}{\partial x} + y'(0)\frac{\partial F}{\partial y} = 1 + 3y'(0). \tag{22}$$

Similarly

$$g(x) = G(x, y(x)) = g(0) + xg'(0) + O(x^2), \tag{23}$$

where

$$g'(0) = \frac{\partial G}{\partial x} + y'(0)\frac{\partial G}{\partial y} = 0. \tag{24}$$

And to first order, the differential term y' near O takes the form

$$y'(x) = y'(0) + O(x). \tag{25}$$

We obtain an indeterminate form where

$$y'(0) = \frac{1 + 3y'(0)}{0}. \tag{26}$$

We may hope that by setting $y'(0) = -1/3$, we can remove the singularity. To see whether this is correct, we have to expand the terms to order 2. We have

$$f''(x) = \frac{\partial^2 F}{\partial x^2} + y''\frac{\partial F}{\partial y} + 2y'\frac{\partial^2 F}{\partial y \partial x} + y'^2\frac{\partial^2 F}{\partial y^2}, \tag{27}$$

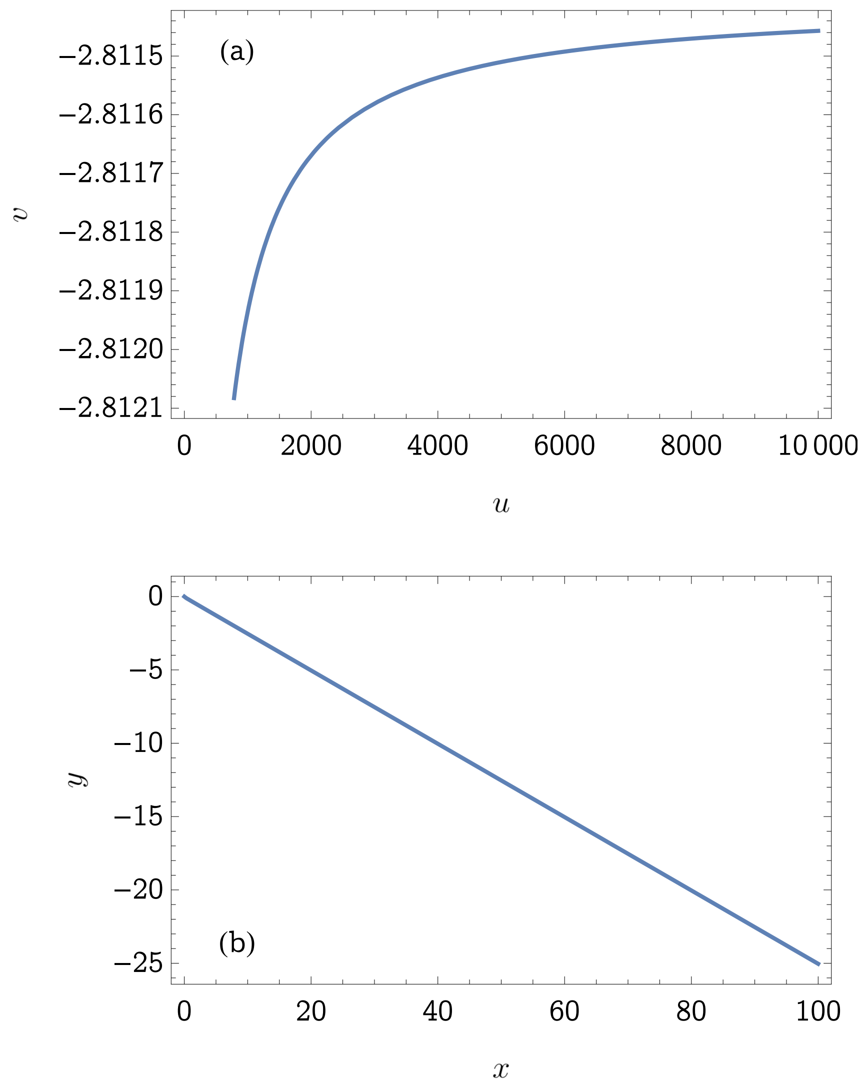

Figure 4: (a) Numerical solution to Eq. (14). (b) Replot of the numerical solution to Eq. (14) in the (x, y) system.

$$f'''(x) = \frac{\partial^3 F}{\partial x^3} + 3y' \frac{\partial^3 F}{\partial x^2 \partial y} + 3y'^2 \frac{\partial^3 F}{\partial x \partial y^2} + 3y'' \frac{\partial^2 F}{\partial y \partial x} + y'^3 \frac{\partial^3 F}{\partial y^3} + 3y'y'' \frac{\partial^2 F}{\partial y^2} + y''' \frac{\partial F}{\partial y}, \tag{28}$$

and

$$g''(x) = \frac{\partial^2 G}{\partial x^2} + y''\frac{\partial G}{\partial y} + 2y'\frac{\partial^2 G}{\partial y \partial x} + y'^2\frac{\partial^2 G}{\partial y^2}. \tag{29}$$

We look for a solution near O in the form

$$y(x) = 0 + mx + \frac{1}{2}px^2 + \frac{1}{6}qx^3 + O(x^4) \tag{30}$$

where $m = y'(0)$, $p = y''(0)$ and $q = y'''(0)$. The derivative is

$$y'(x) = m + px + O(x^2) \tag{31}$$

and the functions F and G become

$$F(x, y(x)) = 0 + (1+3m)x + \frac{1}{2}x^2(6m - 6m^2 + 3p) + \frac{1}{6}x^3(9p - 18mp + 3q), \tag{32}$$

and

$$G(x, y(x)) = 0 + 0x + \frac{1}{2}x^2(12 + 18m) + O(x^3). \tag{33}$$

We thus have to solve

$$\begin{cases} 0 = 1 + 3m, \\ m = \dfrac{6m - 6m^2 + 3p}{12 + 18m}, \\ p = \dfrac{1}{2}\dfrac{9p - 18mp + 3q}{12 + 18m}. \end{cases} \tag{34}$$

We find

$$m = -\frac{1}{3},\ p = \frac{2}{9} \text{ and } q = -\frac{2}{9}. \tag{35}$$

Figure 5 shows the separatrix — the numerical solution to Eq. (14) — and compares it with the Taylor expansion $y = mx + px^2/2 + qx^3/6 + O(x^4)$. Figure 4 shows that $v \to -2.81$ when $u \to \infty$. Recall that this limit found numerically is sensitive to the details of the numerical algorithm used.

Figure 6 shows the phase portrait with the separatrix (Taylor expansion and numerical solution).

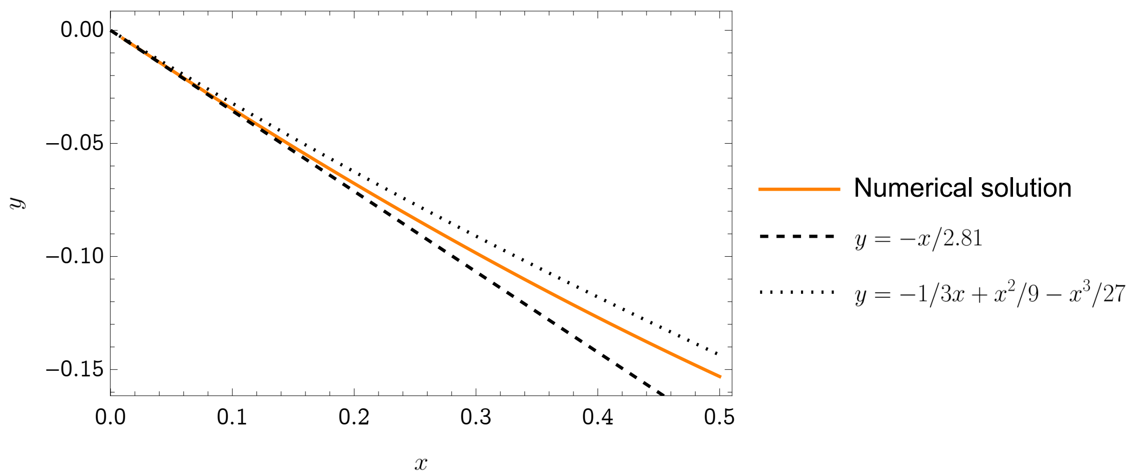

Figure 5: Numerical solution to Eq. (14) with the change of variable (12) and comparison with the Taylor expansion $y = mx + px^2/2 + qx^3/6 + O(x^4)$ — with m, p and q given by Eq. (35) — and $y = -x/2.81$.

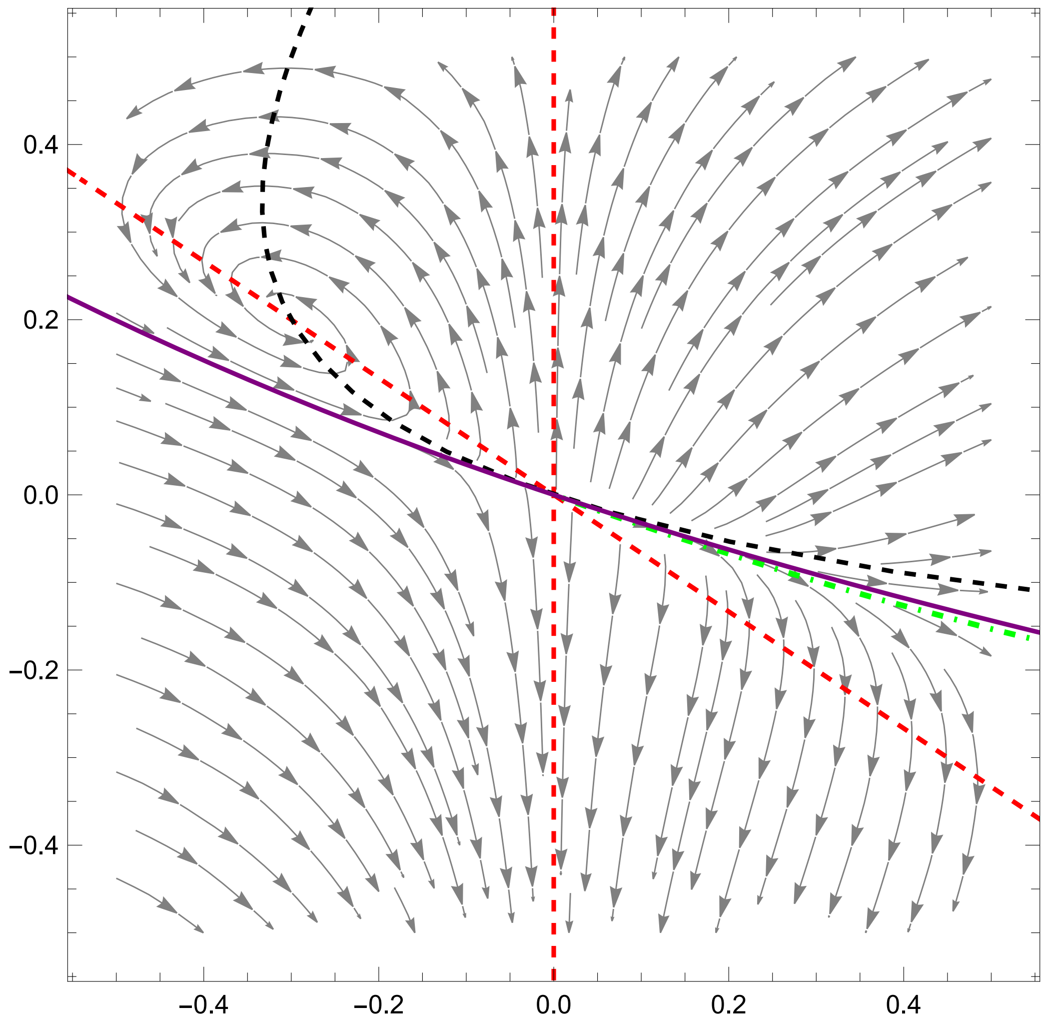

Figure 6: phase portrait of Eq. (1). The dashed black curve shows $F = 0$ while the dashed red lines show the critical curves $G = 0$. The solid line shows the Taylor expansion of the separatrix $y = mx + px^2/2 + qx^3/6 + O(x^4)$, while the dot-dashed green line shows the numerical solution to Eq. (14).